

Parabolas, rates of change and variation

Parabolic antennae

Parallel lines reflected from a parabola will all meet and intersect at one point, called the focal point. A rotated parabola forms the dish shape of parabolic antennae. These antennae are widely used across the world to capture electromagnetic waves, such as TV and radio signals.

For example:

- In global cities with poor reception many residents can access satellite TV signals using a rooftop parabolic antenna with their cable attached to its focal point.
- Australian caravan and boat owners usually carry a parabolic antenna to access VAST or Viewer Access Satellite Television, freely available in areas with weak or no signals.
- In isolated areas of Australia's outback there are mobile hotspots with a phone holder placed at the focal point of the parabolic antenna that receives phone signals.
- Australian scientists use very large parabolic antenna to receive communications from research satellites and outer space, such as NASA's Deep Space Tracking Station located at Tidbinbilla, Canberra.

Chapter contents

- 7A Exploring parabolas
- 7B Sketching parabolas using transformations
- 7C Sketching parabolas using factorisation
- 7D Sketching parabolas by completing the square
- 7E Sketching parabolas using the quadratic formula and the discriminant
- 7F Applications of parabolas
- 7G Intersection of lines and parabolas
- 7H Rates of change
- 7I Average and instantaneous rates of change (EXTENDING)
- 7J Direct variation and inverse variation

NSW Syllabus

In this chapter, a student:

- develops understanding and fluency in mathematics through exploring and connecting mathematical concepts, choosing and applying mathematical techniques to solve problems, and communicating their thinking and reasoning coherently and clearly (MAO-WM-01)
- identifies connections between algebraic and graphical representations of quadratic and exponential relationships in various contexts (MA5-NLI-C-01)
- identifies and compares features of parabolas and exponential curves in various contexts (MA5-NLI-C-02)
- analyses and constructs graphs relating to rates of change (MA5-RAT-P-02)
- identifies and solves problems involving direct and inverse variation and their graphical representations (MA5-RAT-P-01)

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Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

7A Exploring parabolas

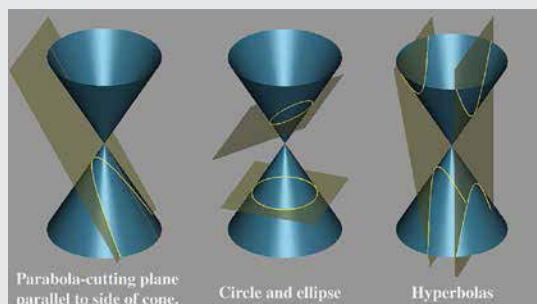
Learning intentions for this section:

- To know the shape and symmetry of the basic parabola
- To be able to identify the key features of a parabola from a graph
- To be able to observe the impact of transformations of the basic parabola

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

One of the simplest and most important non-linear graphs is the parabola. When a ball is thrown or water streams up and out from a garden hose, the path followed has a parabolic shape. The parabola is the graph of a quadratic relation with the basic rule $y = x^2$. Quadratic rules, such as $y = (x - 1)^2$, $y = 2x^2 - x - 3$ and $y = (x + 4)^2 - 7$, also give graphs that are parabolas and are transformations of the graph of $y = x^2$.



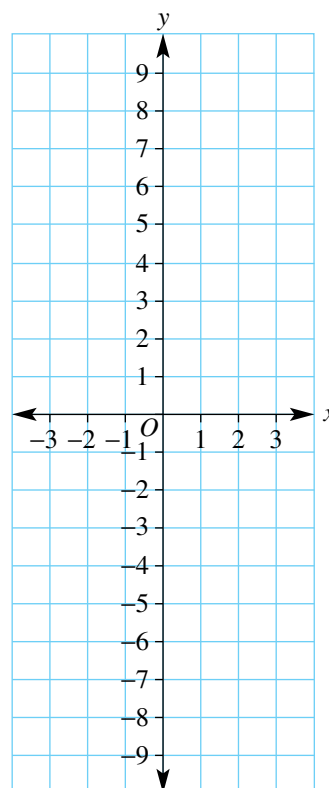
Intersecting a cone with a plane forms curves called the conic sections. Greek scholars analysed the conics using geometry. From the 17th century, Descartes' newly discovered Cartesian geometry enabled a more powerful algebraic analysis.

Lesson starter: To what effect?

To see how different quadratic rules compare to the graph of $y = x^2$, complete this table and plot the graph of each equation on the same set of axes.

x	-3	-2	-1	0	1	2	3
$y_1 = x^2$	9	4					
$y_2 = -x^2$	-9						
$y_3 = (x - 2)^2$							
$y_4 = x^2 - 3$							

- For all the graphs, find such features as the:
 - turning point
 - axis of symmetry
 - y-intercept
 - x-intercepts.
- Discuss how each of the graphs of y_2 , y_3 and y_4 compare to the graph of $y = x^2$. Compare the rule with the position of the graph.



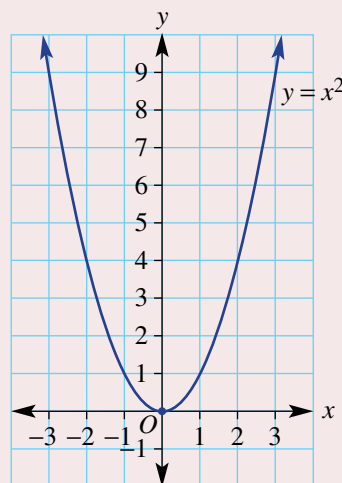
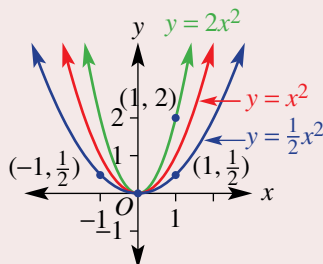
KEY IDEAS

■ A **parabola** is the graph of a quadratic relation. The basic parabola has the rule $y = x^2$.

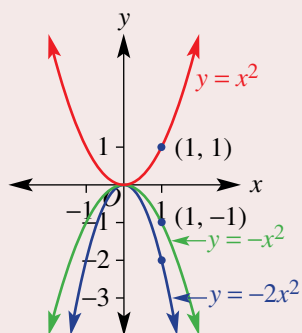
- The vertex (or turning point) is $(0, 0)$.
- It is a minimum turning point.
- Axis of symmetry is $x = 0$.
- y -intercept has coordinates $(0, 0)$.
- x -intercept has coordinates $(0, 0)$.

■ Simple transformations of the graph of $y = x^2$ include:

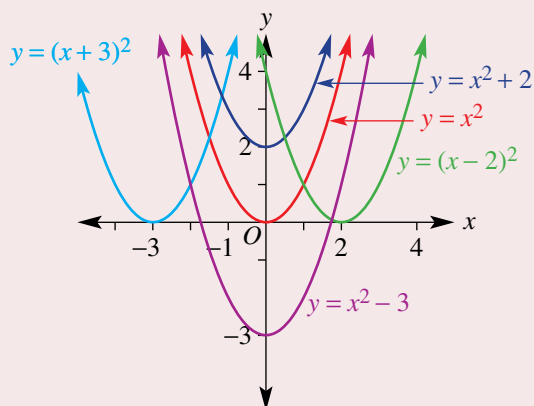
- dilation



- reflection



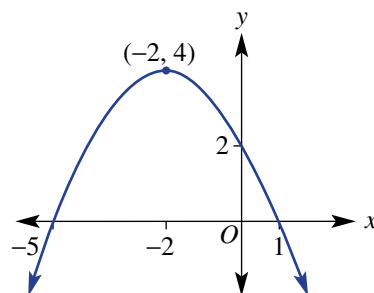
- translation



BUILDING UNDERSTANDING

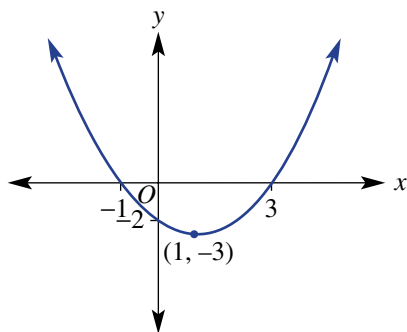
1 Complete the features of this graph.

- The parabola has a _____ (*maximum* or *minimum*).
- The coordinates of the turning point are _____.
- The y -intercept coordinates are $(0, \underline{\hspace{1cm}})$.
- The x -intercepts are at $(\underline{\hspace{1cm}}, 0)$ and $(\underline{\hspace{1cm}}, 0)$.
- The axis of symmetry is _____.



2 Complete the features of this graph.

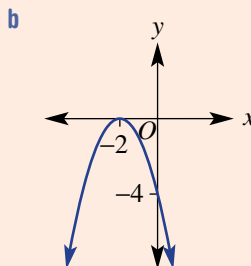
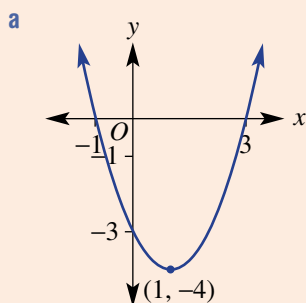
- a The parabola has a _____ (*maximum* or *minimum*).
- b The coordinates of the turning point are _____.
- c The y-intercept coordinates are $(0, \underline{\hspace{1cm}})$.
- d The x-intercepts are $(\underline{\hspace{1cm}}, 0)$ and $(\underline{\hspace{1cm}}, 0)$.
- e The axis of symmetry is _____.



Example 1 Identifying key features of parabolas

Determine the following key features of each of the given graphs.

- i turning point and whether it is a maximum or minimum
- ii axis of symmetry
- iii x-intercept coordinates
- iv y-intercept coordinates



SOLUTION

- a
- i Turning point is a minimum at $(1, -4)$.
 - ii Axis of symmetry is $x = 1$.
 - iii x-intercepts are at $(-1, 0)$ and $(3, 0)$.
 - iv y-intercept is at $(0, -3)$.
- b
- i Turning point is a maximum at $(-2, 0)$.
 - ii Axis of symmetry is $x = -2$.
 - iii x-intercept is at $(-2, 0)$.
 - iv y-intercept is at $(0, -4)$.

EXPLANATION

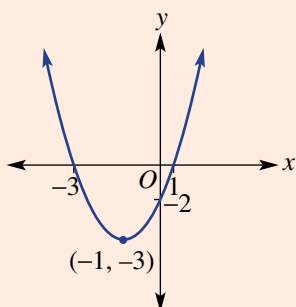
Lowest point of graph is at $(1, -4)$.
 Line of symmetry is through the x -coordinate of the turning point.
 x -intercepts lie on the x -axis ($y = 0$) and the y -intercept on the y -axis ($x = 0$).
 Graph has a highest point at $(-2, 0)$.
 Line of symmetry is through the x -coordinate of the turning point.
 Turning point is also the one x -intercept.

Now you try

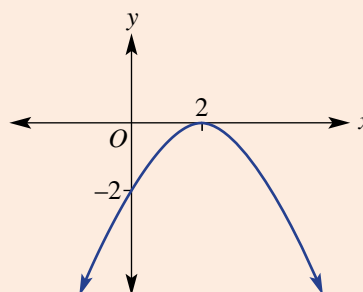
Determine the following key features of each of the given graphs.

- i turning point and whether it is a maximum or minimum
 ii axis of symmetry iii x -intercept coordinates iv y -intercept coordinates

a



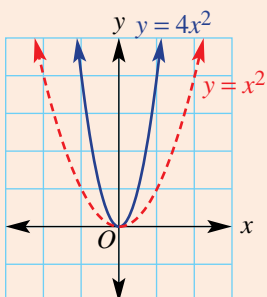
b



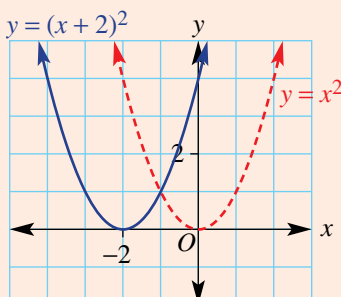
Example 2 Transforming parabolas

Copy and complete the table for the following graphs.

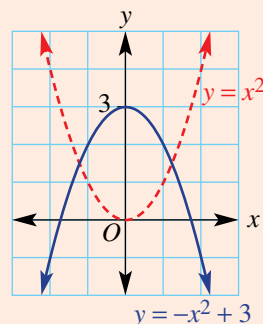
a



b



c



	Formula	Maximum or minimum	Reflected in the x -axis (yes/no)	Turning point	y -value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$					
b	$y = (x + 2)^2$					
c	$y = -x^2 + 3$					

SOLUTION

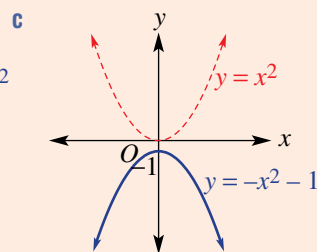
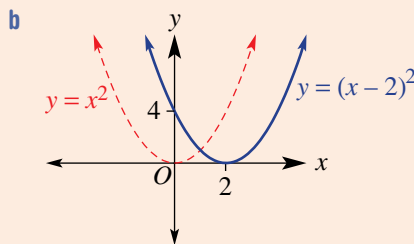
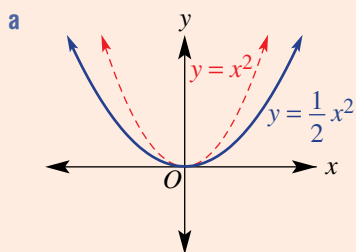
	Formula	Maximum or minimum	Reflected in the x -axis (yes/no)	Turning point	y -value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$	minimum	no	(0, 0)	4	narrower
b	$y = (x + 2)^2$	minimum	no	(-2, 0)	9	same
c	$y = -x^2 + 3$	maximum	yes	(0, 3)	2	same

EXPLANATION

Read features from graphs and consider the effect of each change in equation on the graph.

Now you try

Copy and complete the table for the following graphs.



	Formula	Maximum or minimum	Reflected in the x -axis (yes/no)	Turning point	y -value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = \frac{1}{2}x^2$					
b	$y = (x - 2)^2$					
c	$y = -x^2 - 1$					

Exercise 7A

FLUENCY

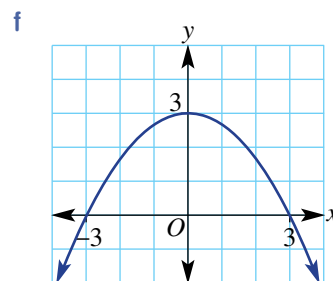
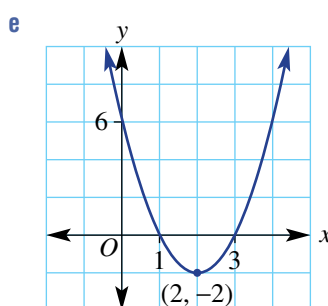
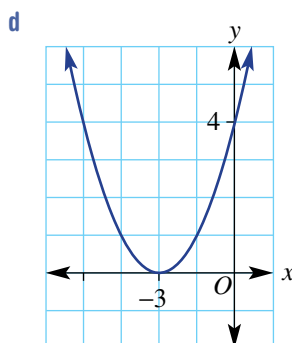
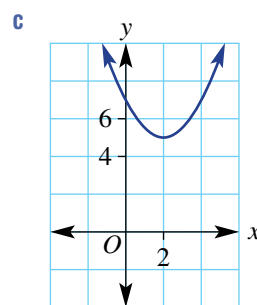
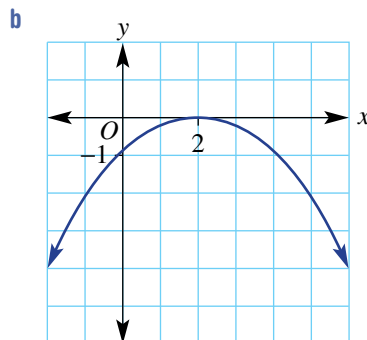
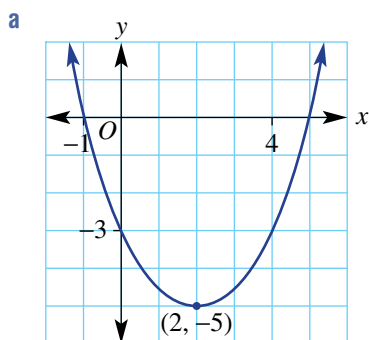
1–4

1–4($\frac{1}{2}$)2–4($\frac{1}{2}$)

Example 1

1 Determine these key features of the following graphs.

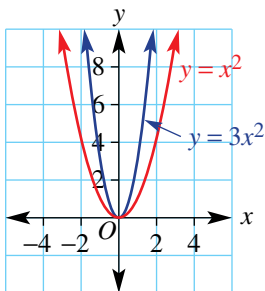
- i turning point and whether it is a maximum or minimum
 ii axis of symmetry
 iii x -intercept coordinates

iv y -intercept coordinates

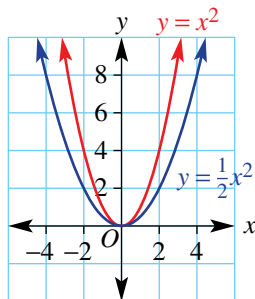
Example 2a

2 Copy and complete the table below for the following graphs.

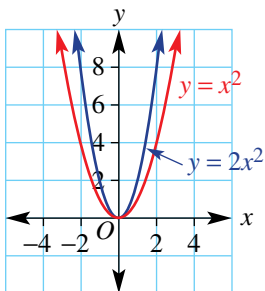
a



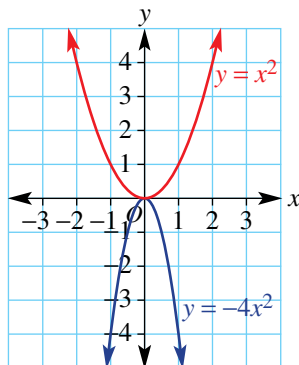
b



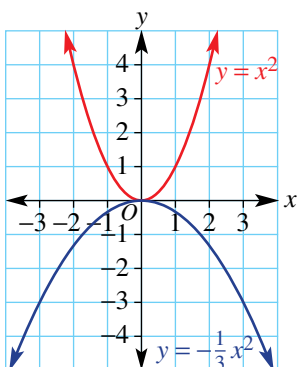
c



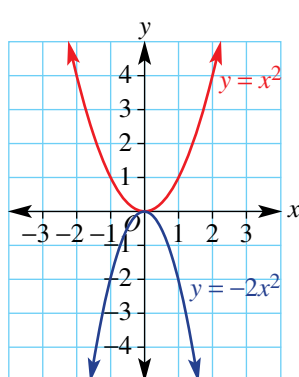
d



e



f

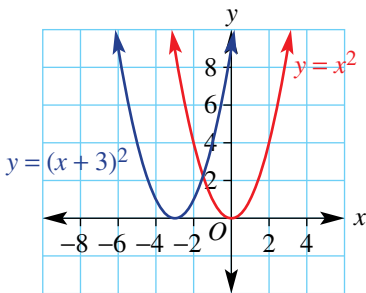


	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$					
b	$y = \frac{1}{2}x^2$					
c	$y = 2x^2$					
d	$y = -4x^2$					
e	$y = -\frac{1}{3}x^2$					
f	$y = -2x^2$					

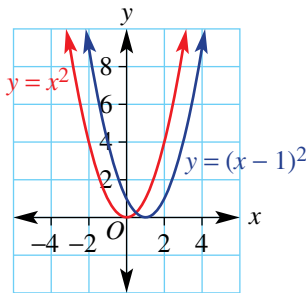
Example 2b

3 Copy and complete the table below for the following graphs.

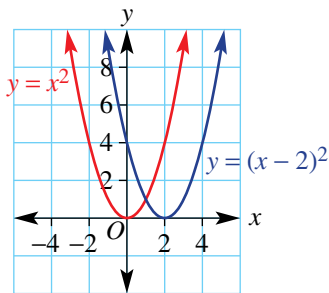
a



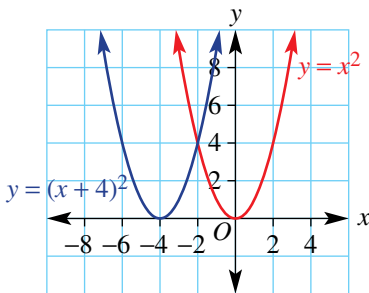
b



c



d

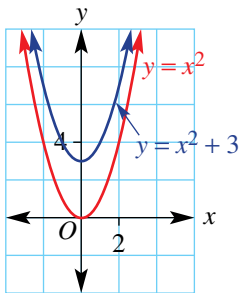


	Formula	Turning point	Axis of symmetry	y-intercept coordinates ($x = 0$)	x-intercept coordinates
a	$y = (x + 3)^2$				
b	$y = (x - 1)^2$				
c	$y = (x - 2)^2$				
d	$y = (x + 4)^2$				

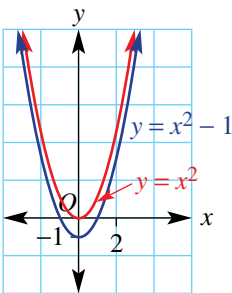
Example 2c

4 Copy and complete the table for the following graphs.

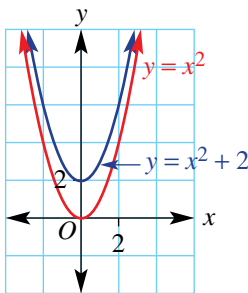
a



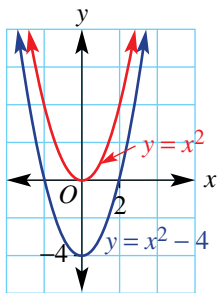
b



c



d



	Formula	Turning point	y-intercept coordinates ($x = 0$)	y-value when $x = 1$
a	$y = x^2 + 3$			
b	$y = x^2 - 1$			
c	$y = x^2 + 2$			
d	$y = x^2 - 4$			

PROBLEM-SOLVING

5–7($\frac{1}{2}$)5–7($\frac{1}{2}$), 85–7($\frac{1}{3}$), 8

5 Write down the equation of the axis of symmetry for the graphs of these rules.

a $y = x^2$

b $y = x^2 + 7$

c $y = -2x^2$

d $y = -3x^2$

e $y = x^2 - 4$

f $y = (x - 2)^2$

g $y = (x + 1)^2$

h $y = -(x + 3)^2$

i $y = -x^2 - 3$

j $y = \frac{1}{2}x^2 + 2$

k $y = x^2 - 16$

l $y = -(x + 4)^2$

6 Write down the coordinates of the turning point for the graphs of the equations in Question 5.

7 Find the coordinates of the y-intercept (i.e. when $x = 0$) for the graphs of the equations in Question 5.

8 Match each of the following equations to one of the graphs below.

a $y = 2x^2$

b $y = x^2 - 6$

c $y = (x + 2)^2$

d $y = -5x^2$

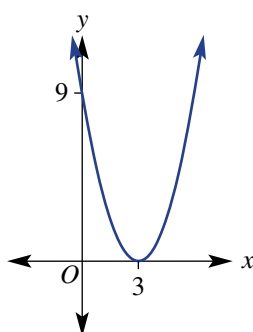
e $y = (x - 3)^2$

f $y = \frac{1}{4}x^2$

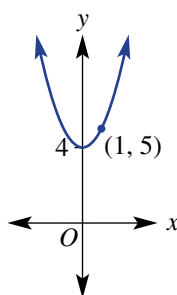
g $y = x^2 + 4$

h $y = (x - 2)^2 + 3$

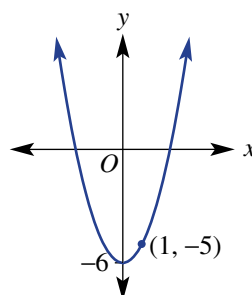
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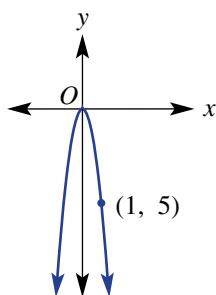
B



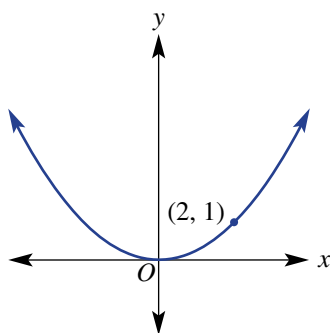
C



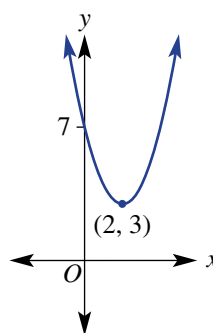
D



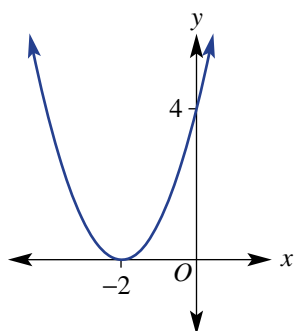
E



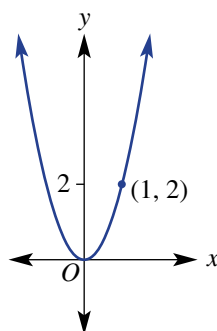
F



G



H



REASONING

9–11

9–11

9–11, 12

- 9 a** Using technology, plot the following pairs of graphs on the same set of axes for $-5 \leq x \leq 5$ and compare their tables of values.
- i** $y = x^2$ and $y = 4x^2$
 - ii** $y = x^2$ and $y = \frac{1}{3}x^2$
 - iii** $y = x^2$ and $y = 6x^2$
 - iv** $y = x^2$ and $y = \frac{1}{4}x^2$
 - v** $y = x^2$ and $y = 7x^2$
 - vi** $y = x^2$ and $y = \frac{2}{5}x^2$
- b** Suggest how the constant a in $y = ax^2$ transforms the graph of $y = x^2$.
- 10 a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leq x \leq 5$ and compare the turning point of each.
- i** $y = x^2$, $y = (x + 1)^2$, $y = (x + 2)^2$, $y = (x + 3)^2$
 - ii** $y = x^2$, $y = (x - 1)^2$, $y = (x - 2)^2$, $y = (x - 3)^2$
- b** Explain how the constant h in $y = (x + h)^2$ transforms the graph of $y = x^2$.
- 11 a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leq x \leq 5$ and compare the turning point of each.
- i** $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 + 3$
 - ii** $y = x^2$, $y = x^2 - 1$, $y = x^2 - 3$, $y = x^2 - 5$
- b** Explain how the constant k in $y = x^2 + k$ transforms the graph of $y = x^2$.
- 12** Write down an example of a quadratic equation whose graph has:
- a** two x -intercepts
 - b** one x -intercept
 - c** no x -intercepts.

ENRICHMENT: Finding the rule

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 $13\frac{1}{2}$, 14

- 13** Find a quadratic rule that satisfies the following information.
- a** turning point (0, 2) and another point (1, 3)
 - b** turning point (0, 2) and another point (1, 1)
 - c** turning point (–1, 0) and y -intercept (0, 1)
 - d** turning point (2, 0) and y -intercept (0, 4)
 - e** turning point (0, 0) and another point (2, 8)
 - f** turning point (0, 0) and another point (–1, –3)
 - g** turning point (–1, 2) and y -intercept (0, 3)
 - h** turning point (4, –2) and y -intercept (0, 0)
- 14** Plot a graph of the parabola $x = y^2$ for $-3 \leq y \leq 3$ and describe its features.



This parabolic solar power collecting array is such that a fluid is heated by the sun and its heat is converted into electricity.

7B Sketching parabolas using transformations

Learning intentions for this section:

- To know the types of transformations: dilation, reflection and translation
- To understand the effect of these transformations on the basic parabola
- To know how to determine the turning point of a quadratic rule from turning point form
- To be able to sketch a quadratic from turning point form
- To be able to find the rule of a quadratic graph given the turning point and another point

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

Previously we have explored simple transformations of the graph of $y = x^2$ and plotted these on a number plane. We will now formalise these transformations and sketch graphs showing key features without the need to plot every point.



Parabolic flight paths occur in athletic jumping and throwing events. Using photography and parabola transformations, sports scientists can find quadratic equations for specific trajectories. Comparing actual and ideal parabolas may reveal areas for technique improvement.

Lesson starter: So where is the turning point?

Consider the quadratic rule $y = -(x - 3)^2 + 7$.

- Discuss the effect of the negative sign in $y = -x^2$ compared with $y = x^2$.
- Discuss the effect of -3 in $y = (x - 3)^2$ compared with $y = x^2$.
- Discuss the effect of $+7$ in $y = x^2 + 7$ compared with $y = x^2$.
- Now for $y = -(x - 3)^2 + 7$, find:
 - the coordinates of the turning point
 - the axis of symmetry
 - the y-intercept.
- What would be the coordinates of the turning point in these quadratics?
 - $y = (x - h)^2 + k$
 - $y = -(x - h)^2 + k$

KEY IDEAS

■ To sketch a parabola, draw a parabolic curve and label key features including:

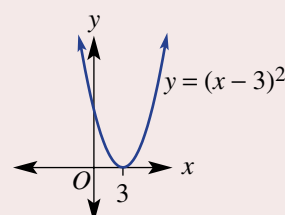
- turning point
- axis of symmetry
- y-intercept (substitute $x = 0$).

■ For $y = ax^2$, a **dilates** the graph of $y = x^2$.

- Turning point is $(0, 0)$.
- y-intercept and x-intercept are both at $(0, 0)$.
- Axis of symmetry is $x = 0$.
- When $a > 0$, the parabola is **upright**.
- When $a < 0$, the parabola is **inverted**.

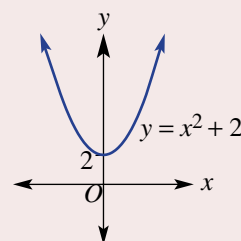
■ For $y = (x - h)^2$, h **translates** the graph of $y = x^2$ horizontally.

- When $h > 0$, the graph is translated h units to the right.
- When $h < 0$, the graph is translated h units to the left.



■ For $y = x^2 + k$, k translates the graph of $y = x^2$ vertically.

- When $k > 0$, the graph is translated k units up.
- When $k < 0$, the graph is translated k units down.



■ The **turning point form** of a quadratic is $y = a(x - h)^2 + k$.

- The turning point is (h, k) .
- The axis of symmetry is $x = h$.

BUILDING UNDERSTANDING

1 Give the coordinates of the turning point for the graphs of these rules.

a $y = x^2$

b $y = x^2 + 3$

c $y = -x^2 - 4$

d $y = (x - 2)^2$

e $y = (x + 5)^2$

f $y = -\frac{1}{3}x^2$

2 Substitute $x = 0$ to find the coordinates of the y-intercept of the graphs with these equations.

a $y = x^2 + 5$

b $y = -x^2 - 3$

c $y = (x + 2)^2$

d $y = (x + 1)^2 + 1$

3 Choose the word: *left*, *right*, *up* or *down* to suit.

a Compared with the graph of $y = x^2$, the graph of $y = x^2 + 3$ is translated _____.

b Compared with the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is translated _____.

c Compared with the graph of $y = x^2$, the graph of $y = (x + 1)^2$ is translated _____.

d Compared with the graph of $y = x^2$, the graph of $y = x^2 - 6$ is translated _____.

e Compared with the graph of $y = -x^2$, the graph of $y = -x^2 - 2$ is translated _____.

f Compared with the graph of $y = -x^2$, the graph of $y = -(x + 3)^2$ is translated _____.

g Compared with the graph of $y = -x^2$, the graph of $y = -(x - 2)^2$ is translated _____.

h Compared with the graph of $y = -x^2$, the graph of $y = -x^2 + 4$ is translated _____.



Example 3 Sketching with transformations

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

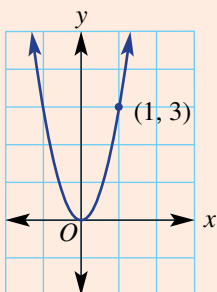
a $y = 3x^2$

b $y = -x^2 + 4$

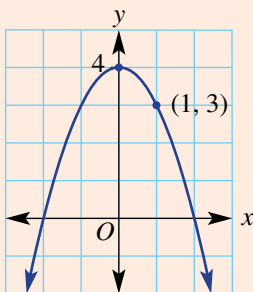
c $y = (x - 2)^2$

SOLUTION

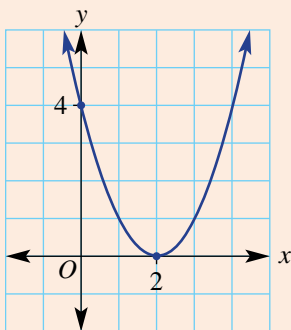
a



b



c



EXPLANATION

$y = 3x^2$ is upright and narrower than $y = x^2$. The turning point and y-intercept are at the origin $(0, 0)$. Substitute $x = 1$ to label a second point.

$y = -x^2 + 4$ is inverted (i.e. has a maximum) and is translated 4 units up compared with $y = -x^2$. The turning point is at $(0, 4)$ and the y-intercept (i.e. when $x = 0$) is also at $(0, 4)$.

Substitute $x = 1$ to label a second point: $y = -1^2 + 4 = 3$.

$y = (x - 2)^2$ is upright (i.e. has a minimum) and is translated 2 units right compared with $y = x^2$. Thus, the turning point is at $(2, 0)$.

Substitute $x = 0$ for the y-intercept: $y = (0 - 2)^2$
 $= (-2)^2$
 $= 4$

The y-intercept is at $(0, 4)$.

Now you try

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

a $y = 2x^2$

b $y = -x^2 + 3$

c $y = (x + 1)^2$



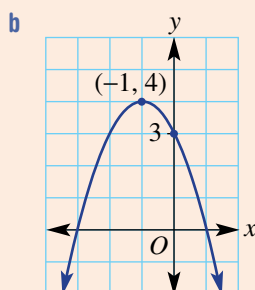
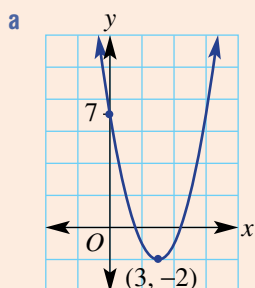
Example 4 Using turning point form

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x - 3)^2 - 2$

b $y = -(x + 1)^2 + 4$

SOLUTION



EXPLANATION

In $y = a(x - h)^2 + k$, $h = 3$ and $k = -2$, so the turning point is $(3, -2)$.

Substitute $x = 0$ to find the y-intercept:

$$\begin{aligned} y &= (0 - 3)^2 - 2 \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

The y-intercept is at $(0, 7)$.

The graph is inverted since $a = -1$.

$h = -1$ and $k = 4$, so the turning point is $(-1, 4)$.

$$\begin{aligned} \text{When } x = 0: y &= -(0 + 1)^2 + 4 \\ &= -1 + 4 \\ &= 3 \end{aligned}$$

The y-intercept is at $(0, 3)$.

Now you try

Sketch the graphs of the following, labelling the turning point and the y-intercept.

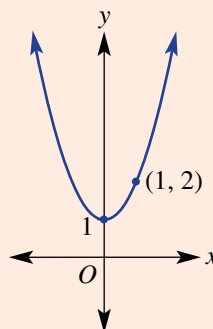
a $y = (x + 1)^2 - 2$

b $y = -(x - 2)^2 + 3$



Example 5 Finding a rule from a simple graph

Determine the rule for this parabola with turning point $(0, 1)$ and another point $(1, 2)$.



SOLUTION

$$y = ax^2 + 1$$

When $x = 1$, $y = 2$ so:

$$2 = a(1)^2 + 1$$

$$\therefore a = 1$$

$$\text{So } y = x^2 + 1.$$

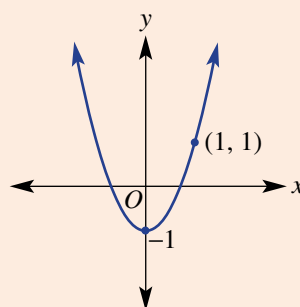
EXPLANATION

In $y = a(x - h)^2 + k$, $h = 0$ and $k = 1$ so the rule is $y = ax^2 + 1$.

We need $y = 2$ when $x = 1$, so $a = 1$.

Now you try

Determine the rule for this parabola with turning point $(0, -1)$ and another point $(1, 1)$.

**Exercise 7B****FLUENCY**

$1, 2-4(\frac{1}{2})$

$1-4(\frac{1}{2})$

$1-4(\frac{1}{3})$

Example 3a

- 1 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept. If the turning point is on the y-axis, also label the point where $x = 1$.

a $y = 2x^2$

b $y = -3x^2$

c $y = \frac{1}{2}x^2$

d $y = -\frac{1}{3}x^2$

Example 3b, c

- 2 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept. If the turning point is on the y-axis, also label the point where $x = 1$.

a $y = x^2 + 2$

b $y = x^2 - 4$

c $y = -x^2 + 1$

d $y = -x^2 - 3$

e $y = (x + 3)^2$

f $y = (x - 1)^2$

g $y = -(x + 2)^2$

h $y = -(x - 3)^2$

i $y = (x + 4)^2$

- 3 State the coordinates of the turning point for the graphs of these rules.

a $y = (x + 3)^2 + 1$

b $y = (x + 2)^2 - 4$

c $y = (x - 1)^2 + 3$

d $y = (x - 4)^2 - 2$

e $y = (x - 3)^2 - 5$

f $y = (x - 2)^2 + 2$

g $y = -(x - 3)^2 + 3$

h $y = -(x - 2)^2 + 6$

i $y = -(x + 1)^2 + 4$

j $y = -(x - 2)^2 - 5$

k $y = -(x + 1)^2 - 1$

l $y = -(x - 4)^2 - 10$

Example 4

- 4 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 + 1$

b $y = (x + 2)^2 - 1$

c $y = (x + 3)^2 + 2$

d $y = (x - 1)^2 + 2$

e $y = (x - 4)^2 + 1$

f $y = (x - 1)^2 - 4$

g $y = -(x - 1)^2 + 3$

h $y = -(x - 2)^2 + 1$

i $y = -(x + 3)^2 - 2$

j $y = -(x - 2)^2 + 1$

k $y = -(x - 4)^2 - 2$

l $y = -(x + 2)^2 + 2$

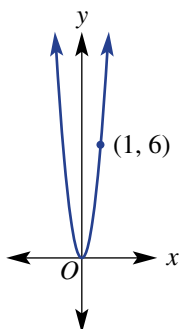
PROBLEM-SOLVING

 $5(\frac{1}{2})$ $5-6(\frac{1}{2})$ $5-6(\frac{1}{3}), 7$

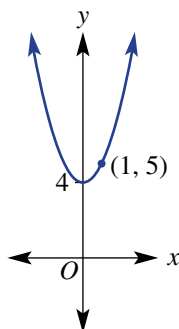
Example 5

5 Determine the rule for the following parabolas.

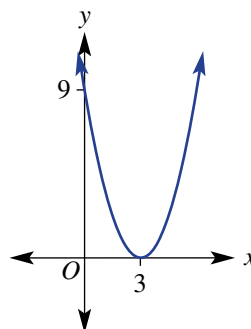
a



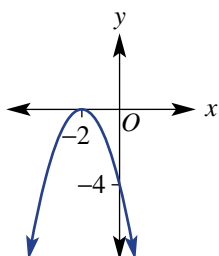
b



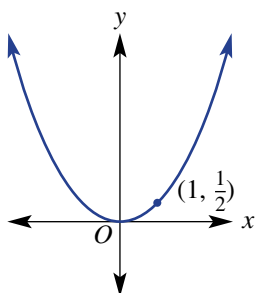
c



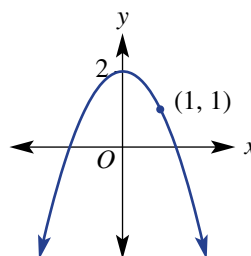
d



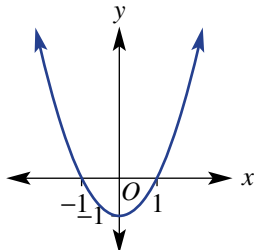
e



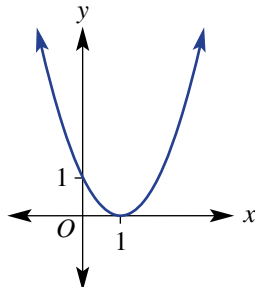
f



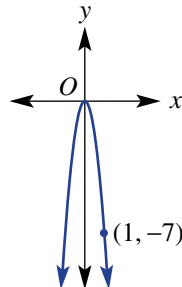
g



h



i

6 Write the rule for each graph when $y = x^2$ is transformed by the following.

- a reflected in the x -axis
- b translated 2 units to the left
- c translated 5 units down
- d translated 4 units up
- e translated 1 unit to the right
- f reflected in the x -axis and translated 2 units up
- g reflected in the x -axis and translated 3 units left
- h translated 5 units left and 3 units down
- i translated 6 units right and 1 unit up

- 7 The path of a basketball is given by $y = -(x - 5)^2 + 25$, where y metres is the height and x metres is the horizontal distance.
- Is the turning point a maximum or a minimum?
 - What are the coordinates of the turning point?
 - What are the coordinates of the y -intercept?
 - What is the maximum height of the ball?
 - What is the height of the ball at these horizontal distances?
 - $x = 3$
 - $x = 7$
 - $x = 10$

REASONING $8(\frac{1}{2})$ $8-9(\frac{1}{2})$ $9(\frac{1}{2}), 10$

- 8 Recall that $y = (x - h)^2 + k$ and $y = a(x - h)^2 + k$ both have the same turning point coordinates. State the coordinates of the turning point for the graphs of these rules.
- | | |
|--------------------------------|--------------------------------|
| a $y = 2(x - 1)^2$ | b $y = 3(x + 2)^2$ |
| c $y = -4(x + 3)^2$ | d $y = 3x^2 - 4$ |
| e $y = 5x^2 - 2$ | f $y = -2x^2 + 5$ |
| g $y = 6(x + 4)^2 - 1$ | h $y = 2(x + 2)^2 + 3$ |
| i $y = 3(x - 5)^2 + 4$ | j $y = -4(x + 2)^2 + 3$ |
| k $y = -2(x + 3)^2 - 5$ | l $y = -5(x - 3)^2 - 3$ |
- 9 Describe the transformations that take $y = x^2$ to:
- | | | |
|------------------------------|---------------------------|------------------------------|
| a $y = (x - 3)^2$ | b $y = (x + 2)^2$ | c $y = x^2 - 3$ |
| d $y = x^2 + 7$ | e $y = -x^2$ | f $y = (x + 2)^2 - 4$ |
| g $y = (x - 5)^2 + 8$ | h $y = -(x + 3)^2$ | i $y = -x^2 + 6$ |
- 10 For $y = a(x - h)^2 + k$, write:
- | | |
|---|---|
| a the coordinates of the turning point | b the coordinates of the y -intercept. |
|---|---|

ENRICHMENT: Sketching with many transformations

–

–

 $11(\frac{1}{2})$

- 11 Sketch the graph of the following, showing the turning point and the y -intercept.
- | | |
|---|--|
| a $y = 2(x - 3)^2 + 4$ | b $y = 3(x + 2)^2 + 5$ |
| c $y = -2(x - 3)^2 + 4$ | d $y = -2(x + 3)^2 - 4$ |
| e $y = \frac{1}{2}(x - 3)^2 + 4$ | f $y = -\frac{1}{2}(x - 3)^2 + 4$ |
| g $y = 4 - x^2$ | h $y = -3 - x^2$ |
| i $y = 5 - 2x^2$ | j $y = 2 + \frac{1}{2}(x - 1)^2$ |
| k $y = 1 - 2(x + 2)^2$ | l $y = 3 - 4(x - 2)^2$ |

7C Sketching parabolas using factorisation

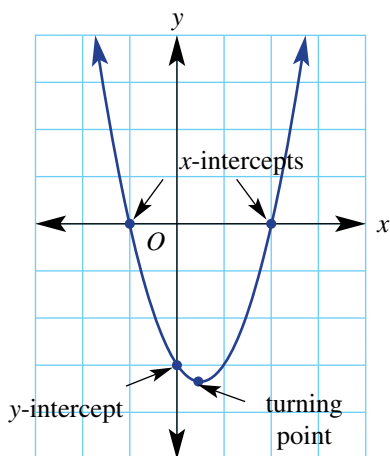
Learning intentions for this section:

- To know the steps for sketching a quadratic graph
- To understand that quadratic graphs can have 0, 1 or 2 x -intercepts
- To know how to use factorisation to determine the x -intercepts of a quadratic graph
- To know how to use symmetry to locate the turning point once the x -intercepts are known

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

A quadratic relation written in the form $y = x^2 + bx + c$ differs from that of turning point form, $y = a(x - h)^2 + k$, and so the transformations of the graph of $y = x^2$ to give $y = x^2 + bx + c$ are less obvious. To address this, we have a number of options. We can first try to factorise to find the x -intercepts and then use symmetry to find the turning point or, alternatively, we can complete the square and express the quadratic relation in turning point form. The second of these methods will be studied in **Section 7D**.



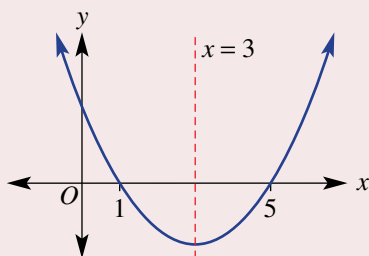
Engineers can develop equations of parabolic shapes, such as the St Louis archway, by reversing the factorisation procedure. A sketch is labelled with measurements, the x -intercepts form the factors and expanding the brackets gives the basic quadratic equation.

Lesson starter: Why does the turning point of $y = x^2 - 2x - 3$ have coordinates $(1, -4)$?

- Factorise $y = x^2 - 2x - 3$.
- Hence, find the x -intercepts.
- Discuss how symmetry can be used to locate the turning point.
- Hence, confirm the coordinates of the turning point.

KEY IDEAS

- To sketch a graph of $y = x^2 + bx + c$:
 - find the y -intercept by substituting $x = 0$
 - find the x -intercept(s) by substituting $y = 0$. Factorise where possible and use the Null Factor Law (if $p \times q = 0$, then $p = 0$ or $q = 0$).
- Once the x -intercepts are known, the turning point can be found using symmetry.
 - The axis of symmetry (also the x -coordinate of the turning point) lies halfway between the x -intercepts: $x = \frac{1+5}{2} = 3$, for the graph below.



- Substitute this x -coordinate into the rule to find the y -coordinate of the turning point.

BUILDING UNDERSTANDING

- 1 Use the Null Factor Law to find the coordinates of the x -intercepts ($y = 0$) for these factorised quadratics.

a $y = (x + 1)(x - 2)$
b $y = x(x - 3)$
c $y = -3x(x + 2)$
- 2 Factorise these quadratics.

a $y = x^2 - 4x$
b $y = x^2 + 2x - 8$

c $y = x^2 - 8x + 16$
d $y = x^2 - 25$
- 3 Find the coordinates of the y -intercept for the quadratics in Question 2.
- 4 Use the given rule and x -intercepts to find the coordinates of the turning point.

a $y = x^2 - 8x + 12$, x -intercepts: at $x = 2$ and $x = 6$
b $y = -x^2 - 2x + 8$, x -intercepts: at $x = -4$ and $x = 2$



Example 6 Using the x -intercepts to find the turning point

Sketch the graph of the following quadratics by using the x -intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 2x$

b $y = x^2 - 6x + 5$

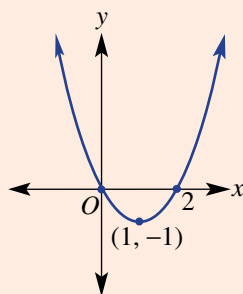
SOLUTION

a y -intercept at $x = 0$: $y = 0$
 x -intercepts at $y = 0$: $0 = x^2 - 2x$
 $0 = x(x - 2)$
 $x = 0$ or $x - 2 = 0$
 $\therefore x = 0, x = 2$

Turning point at $x = \frac{0+2}{2} = 1$.

$$y = 1^2 - 2(1) = -1$$

Turning point is a minimum at $(1, -1)$.



b y -intercept at $x = 0$: $y = 5$
 x -intercepts at $y = 0$: $0 = x^2 - 6x + 5$
 $0 = (x - 5)(x - 1)$
 $x - 5 = 0$ or $x - 1 = 0$
 $\therefore x = 5, x = 1$

Turning point at $x = \frac{1+5}{2} = 3$.

$$y = (3)^2 - 6 \times (3) + 5 = 9 - 18 + 5 = -4$$

Turning point is a minimum at $(3, -4)$.

EXPLANATION

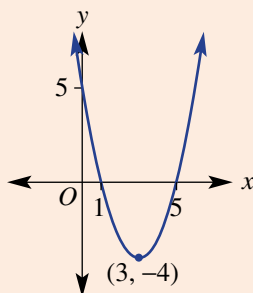
Identify key features of the graph:
 y -intercept (when $x = 0$), x -intercepts (when $y = 0$), then factorise by noting the common factor and solve by applying the Null Factor Law. Recall that if $p \times q = 0$, then $p = 0$ or $q = 0$.

Using symmetry the x -coordinate of the turning point is halfway between the x -coordinates of the x -intercepts.
 Substitute $x = 1$ into $y = x^2 - 2x$ to find the y -coordinate of the turning point.
 It is a minimum turning point since the coefficient of x^2 is positive.

Label key features on the graph and join points in the shape of a parabola.

Identify key features of the graph:
 y -intercept ($y = 0^2 - 6(0) + 5$) and x -intercepts by factorising and applying the Null Factor Law.

Using symmetry the x -coordinate of the turning point is halfway between the x -coordinates of the x -intercepts.
 Substitute $x = 3$ into $y = x^2 - 6x + 5$ to find the y -coordinate of the turning point.
 It is a minimum turning point since the coefficient of x^2 is positive.



Label key features on the graph and join points in the shape of a parabola.

Now you try

Sketch the graph of the following quadratics by using the x -intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 6x$

b $y = x^2 - 8x + 7$



Example 7 Sketching a perfect square

Sketch the graph of the quadratic $y = x^2 + 6x + 9$.

SOLUTION

y-intercept at $x = 0$: $y = 9$

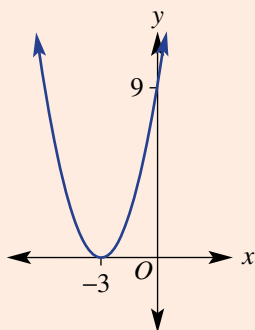
x-intercepts at $y = 0$: $0 = x^2 + 6x + 9$

$$0 = (x + 3)^2$$

$$x + 3 = 0$$

$$\therefore x = -3$$

Turning point is at $(-3, 0)$.



EXPLANATION

For y-intercept substitute $x = 0$.

For x-intercepts substitute $y = 0$ and factorise: $(x + 3)(x + 3) = (x + 3)^2$. Apply the Null Factor Law to solve for x .

As there is only one x -intercept, it is also the turning point.

Label key features on the graph.

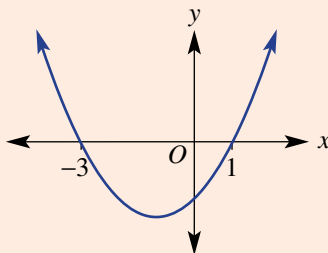
Now you try

Sketch the graph of the quadratic $y = x^2 + 8x + 16$.



Example 8 Finding a turning point from a graph

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x -intercepts to find the values of a and b , then find the coordinates of the turning point.



SOLUTION

$$a = 3 \text{ and } b = -1$$

$$y = (x + 3)(x - 1)$$

$$x\text{-coordinate of the turning point is } \frac{-3 + 1}{2} = -1.$$

$$y\text{-coordinate is } (-1 + 3)(-1 - 1) = 2 \times (-2) = -4.$$

Turning point is $(-1, -4)$.

EXPLANATION

Using the Null Factor Law,

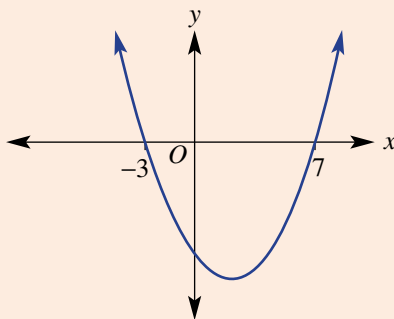
$(x + 3)(x - 1) = 0$ gives $x = -3$ and $x = 1$, so $a = 3$ and $b = -1$.

Find the average of the two x -intercepts to find the x -coordinate of the turning point.

Substitute $x = -1$ into the rule to find the y -value of the turning point.

Now you try

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x -intercepts to find the values of a and b , then find the coordinates of the turning point.



Exercise 7C

FLUENCY

1–4($\frac{1}{2}$)1–5($\frac{1}{2}$)1–5($\frac{1}{3}$)

Example 6a

- 1 Sketch the graph of the following quadratics by using the x -intercepts to help determine the coordinates of the turning point.

a $y = x^2 + 2x$

b $y = x^2 + 6x$

c $y = x^2 - 4x$

d $y = x^2 - 5x$

e $y = x^2 + 3x$

f $y = x^2 + 7x$

Example 6b

- 2 Sketch the graph of the following quadratics by using the x -intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 6x + 8$

b $y = x^2 - 8x + 12$

c $y = x^2 + 8x + 15$

d $y = x^2 - 6x - 16$

e $y = x^2 - 2x - 8$

f $y = x^2 - 4x - 21$

g $y = x^2 + 8x + 7$

h $y = x^2 - 12x + 20$

- 3 Sketch graphs of the following quadratics.

a $y = x^2 - 9x + 20$

b $y = x^2 - 5x + 6$

c $y = x^2 - 13x + 12$

d $y = x^2 + 11x + 30$

e $y = x^2 + 5x + 4$

f $y = x^2 + 13x + 12$

g $y = x^2 - 4x - 12$

h $y = x^2 - x - 2$

i $y = x^2 - 5x - 14$

j $y = x^2 + 3x - 4$

k $y = x^2 + 7x - 30$

l $y = x^2 + 9x - 22$

Example 7

- 4 Sketch graphs of the following perfect squares.

a $y = x^2 + 4x + 4$

b $y = x^2 + 8x + 16$

c $y = x^2 - 10x + 25$

d $y = x^2 + 20x + 100$

- 5 Sketch graphs of the following quadratics that include a difference of two squares.

a $y = x^2 - 9$

b $y = x^2 - 16$

c $y = x^2 - 4$

PROBLEM-SOLVING

6($\frac{1}{2}$), 76($\frac{1}{2}$), 7, 8

7, 9, 10

- 6 Determine the turning points of the following quadratics.

a $y = 2(x^2 - 7x + 10)$

b $y = 3(x^2 - 7x + 10)$

c $y = 3x^2 + 18x + 24$

d $y = 4x^2 + 24x + 32$

e $y = 4(x^2 - 49)$

f $y = -4(x^2 - 49)$

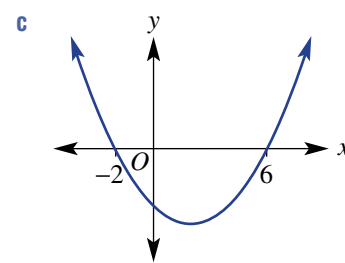
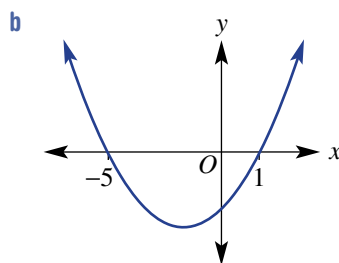
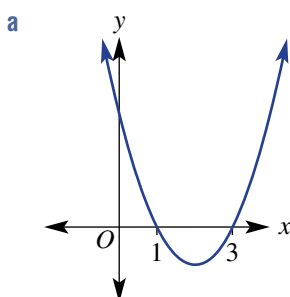
g $y = 3x^2 - 6x + 3$

h $y = 5x^2 - 10x + 5$

i $y = 2x^2 - 4x + 10$

Example 8

- 7 The equations of these graphs are of the form $y = (x + a)(x + b)$. Use the x -intercepts to find the values of a and b , and then find the coordinates of the turning point.



8 State the coordinates of the x -intercepts and turning point for these quadratics.

a $y = x^2 - 2$

b $y = x^2 - 11$

c $y = 2x^2 - 10$

9 Sketch a graph of these quadratics.

a $y = 9 - x^2$

b $y = 1 - x^2$

c $y = 4x - x^2$

d $y = 3x - x^2$

e $y = -x^2 + 2x + 8$

f $y = -x^2 + 8x + 9$

10 If the graph of $y = a(x + 2)(x - 4)$ passes through the point $(2, 16)$, determine the value of a and the coordinates of the turning point for this parabola.

REASONING

11

11, 12

12–14

11 Explain why $y = (x - 3)(x - 5)$ and $y = 2(x - 3)(x - 5)$ both have the same x -intercepts.

12 a Explain why $y = x^2 - 2x + 1$ has only one x -intercept.

b Explain why $y = x^2 + 2$ has zero x -intercepts.

13 Consider the quadratics $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$.

a Show that both quadratics have the same x -intercepts.

b Find the coordinates of the turning points for both quadratics.

c Compare the positions of the turning points.

14 A quadratic has the rule $y = x^2 + bx$. Give the coordinates of:

a the y -intercept

b the x -intercepts

c the turning point.

ENRICHMENT: More rules from graphs

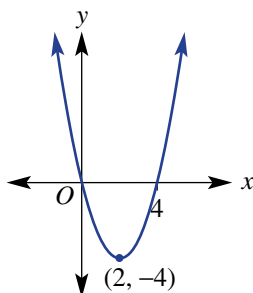
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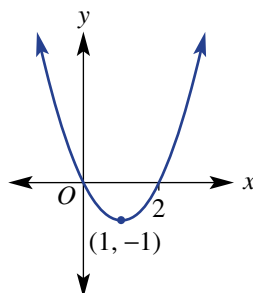
15(½)

15 Determine the equation of each of these graphs in factorised form; for example, $y = 2(x - 3)(x + 2)$.

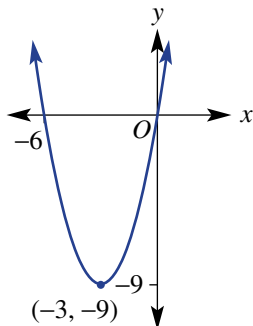
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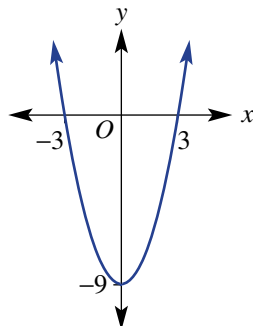
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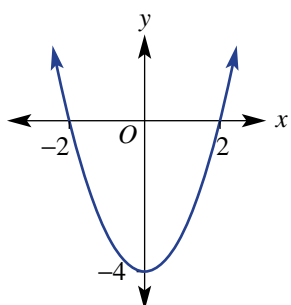
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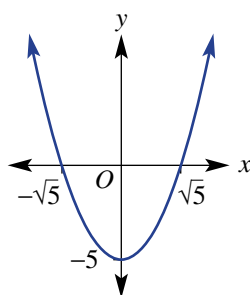
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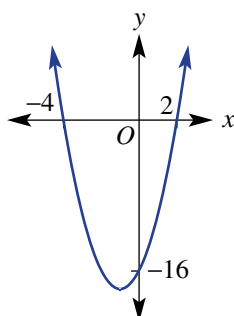
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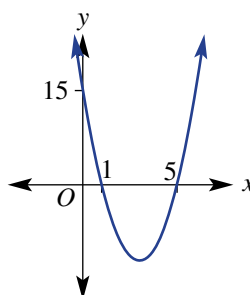
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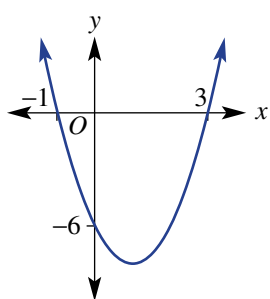
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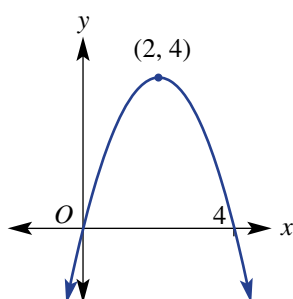
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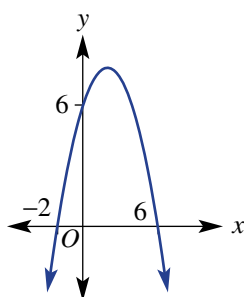
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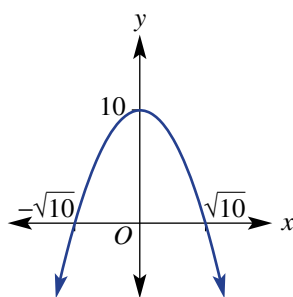
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l



7D Sketching parabolas by completing the square

Learning intentions for this section:

- To know that completing the square can be used to express any quadratic in turning point form
- To be able to find any x -intercepts from the turning point form of a quadratic
- To be able to sketch a quadratic equation in turning point form, labelling key features

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

We have learnt previously that the turning point of a parabola can be read directly from a rule in the form $y = a(x - h)^2 + k$. This form of quadratic rule can be obtained by completing the square.

Lesson starter: I forgot how to complete the square!

To make $x^2 + 6x$ a perfect square we need to add 9
(from $\left(\frac{6}{2}\right)^2$) since $x^2 + 6x + 9 = (x + 3)^2$.

So to complete the square for $x^2 + 6x + 2$ we

write $x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 2 = (x + 3)^2 - 7$.

- Discuss the rules for completing the square and explain how $x^2 + 6x + 2$ becomes $(x + 3)^2 - 7$.
- What does the turning point form of $x^2 + 6x + 2$ tell us about its graph?
- How can you use the turning point form of $x^2 + 6x + 2$ to help find the x -intercepts of $y = x^2 + 6x + 2$?



Businesses use mathematical modelling to analyse profits. Quadratic equations model revenue vs selling price and its graph is an inverted parabola. With rising prices, revenue grows until the turning point, then it decreases due to declining sales.

KEY IDEAS

■ By **completing the square**, all quadratics in the form $y = ax^2 + bx + c$ can be expressed in turning point form; i.e. $y = a(x - h)^2 + k$.

■ To sketch a quadratic in the form $y = a(x - h)^2 + k$, follow these steps.

- Determine the coordinates of the turning point (h, k) .
 - When a is positive, the parabola has a minimum turning point.
 - When a is negative, the parabola has a maximum turning point.
- Determine the y -intercept by substituting $x = 0$.
- Determine the x -intercepts, if any, by substituting $y = 0$ and solving the equation.

■ To solve $x^2 = a$, $a > 0$, take the square root of both sides: $x = \pm\sqrt{a}$. i.e. $\sqrt{a}$ and $-\sqrt{a}$.

For any perfect square, say $(x + 1)^2 = 16$, take the square root of both sides:

$$\begin{aligned} x + 1 &= \pm 4 \\ x &= -1 \pm 4 \\ x &= -1 + 4 \text{ or } x = -1 - 4 \\ x &= 3 \text{ or } x = -5 \end{aligned}$$

BUILDING UNDERSTANDING

- 1 By completing the square, state the coordinates of the turning point (TP).

a $y = x^2 + 2x - 5$
 $= x^2 + 2x + \underline{\hspace{1cm}} - \underline{\hspace{1cm}} - \underline{\hspace{1cm}}$
 $= (\underline{\hspace{1cm}})^2 - \underline{\hspace{1cm}}$
 TP = ($\underline{\hspace{1cm}}$, $\underline{\hspace{1cm}}$)

b $y = x^2 - 6x + 10$
 $= x^2 - 6x + \underline{\hspace{1cm}} - \underline{\hspace{1cm}} + \underline{\hspace{1cm}}$
 $= \underline{\hspace{1cm}}$
 TP = ($\underline{\hspace{1cm}}$, $\underline{\hspace{1cm}}$)

- 2 Solve these equations for x , giving exact answers.

a $x^2 = 9$

b $x^2 = 3$

c $(x - 1)^2 = 16$

d $(x + 4)^2 = 2$



Example 9 Finding key features of quadratics in turning point form

For $y = -4(x - 1)^2 + 16$:

- a determine the coordinates of its turning point and state whether it is a maximum or minimum
 b determine the coordinates of the y -intercept
 c determine the coordinates of the x -intercepts (if any).

SOLUTION

- a Turning point is a maximum at $(1, 16)$.

- b y -intercept at $x = 0$:

$$\begin{aligned} y &= -4(0 - 1)^2 + 16 \\ &= -4 + 16 \\ &= 12 \end{aligned}$$

$\therefore y$ -intercept is at $(0, 12)$.

- c x -intercepts at $y = 0$:

$$\begin{aligned} 0 &= -4(x - 1)^2 + 16 \\ 0 &= (x - 1)^2 - 4 \\ (x - 1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= 1 \pm 2 \\ x &= -1, 3 \end{aligned}$$

$\therefore x$ -intercepts are at $(-1, 0)$ and $(3, 0)$.

EXPLANATION

For $y = a(x - h)^2 + k$ the turning point is at (h, k) . As $a = -4$ is negative, the parabola has a maximum turning point.

Substitute $x = 0$ to find the y -intercept. Recall that $(0 - 1)^2 = (-1)^2 = 1$.

Substitute $y = 0$ for x -intercepts.

Divide both sides by -4 : $16 \div (-4) = -4$.

Add 4 to both sides, and take the square root of both sides.

Answers are ± 2 since $2^2 = 4$ and $(-2)^2 = 4$.

$1 - 2 = -1$ and $1 + 2 = 3$ are the x -intercepts.

Note: Check that the x -intercepts are evenly spaced either side of the turning point.

Alternatively, use difference of two squares to write in factorised form:

$$(x - 1)^2 - 2^2 = 0$$

$$(x - 1 - 2)(x - 1 + 2) = 0$$

and apply the Null Factor Law to solve for x .

Now you try

For $y = -2(x + 1)^2 + 18$:

- a** determine the coordinates of its turning point and state whether it is a maximum or minimum
- b** determine the coordinates of the y -intercept
- c** determine the coordinates of the x -intercepts (if any).

**Example 10 Sketching by completing the square**

Sketch these graphs by completing the square, giving the x -intercepts in exact form.

a $y = x^2 + 6x + 15$

b $y = x^2 - 4x + 2$

c $y = x^2 - 3x - 1$

SOLUTION

- a** Turning point form:

$$\begin{aligned} y &= x^2 + 6x + 15 \\ &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 15 \\ &= (x + 3)^2 + 6 \end{aligned}$$

Turning point is a minimum at $(-3, 6)$.

y -intercept at $x = 0$:

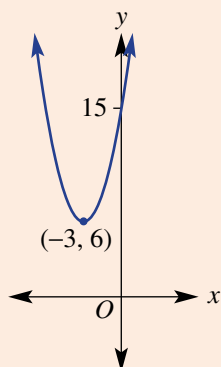
$$\begin{aligned} y &= (0)^2 + 6(0) + 15 \\ &= 15 \end{aligned}$$

$\therefore$ y -intercept is at $(0, 15)$.

x -intercepts at $y = 0$:

$$0 = (x + 3)^2 + 6$$

There is no solution and there are no x -intercepts.

**EXPLANATION**

To change the equation into turning point form, complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$.

Read off the turning point, which is a minimum, as $a = 1$ is positive.

For the y -intercept, substitute $x = 0$ into the original equation.

For the x -intercepts, substitute $y = 0$ into the turning point form.

This cannot be solved as $(x + 3)^2$ cannot equal -6 , hence there are no x -intercepts. Note also that the turning point is a minimum with a lowest y -coordinate of 6, telling us there are no x -intercepts.

Sketch the graph, showing the key points.

b Turning point form: $y = x^2 - 4x + 2$

$$y = x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 + 2$$

$$= (x - 2)^2 - 2$$

Turning point is a minimum at $(2, -2)$.

y -intercept at $x = 0$:

$$y = 0^2 - 4(0) + 2$$

$$= 2$$

$\therefore y$ -intercept is at $(0, 2)$.

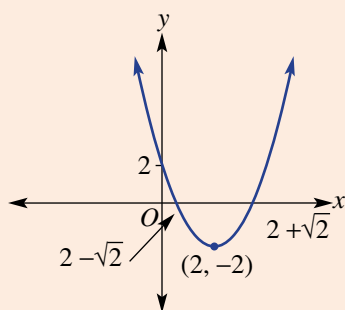
x -intercepts at $y = 0$:

$$0 = (x - 2)^2 - 2$$

$$\therefore (x - 2)^2 = 2$$

$$x - 2 = \pm\sqrt{2}$$

$$x = 2 \pm \sqrt{2}$$



c Turning point form:

$$y = x^2 - 3x - 1$$

$$= x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 1$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

Turning point is a minimum at $\left(\frac{3}{2}, -\frac{13}{4}\right)$.

y -intercept at $x = 0$:

$$y = (0)^2 - 3(0) - 1$$

$$= -1$$

$\therefore y$ -intercept is at $(0, -1)$.

x -intercepts at $y = 0$:

$$0 = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{13}{4}$$

$$x - \frac{3}{2} = \pm\frac{\sqrt{13}}{2}$$

$$x = \frac{3 + \sqrt{13}}{2}, x = \frac{3 - \sqrt{13}}{2}$$

Complete the square to express the rule in turning point form.

Read off the turning point, which is a minimum since the coefficient of x^2 is positive.

Substitute $x = 0$ to find the y -intercept.

Substitute $y = 0$ to find the x -intercepts and solve the resulting equation.

Add 2 to both sides of the equation and take the square root of both sides, remembering to include $\pm$.

Sketch the graph, labelling key points.

Note that the x -intercepts are positioned symmetrically either side of the turning point.

Complete the square to write in

turning point form: $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ and

$$-\frac{9}{4} - 1 = -\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}.$$

Substitute $x = 0$ to find the y -intercept.

Substitute $y = 0$ to find the x -intercepts.

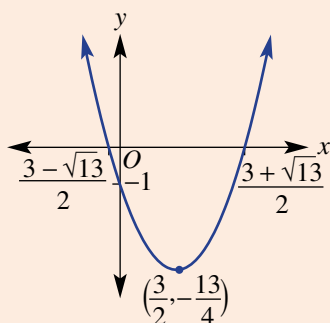
Add $\frac{13}{4}$ to both sides and take the square root.

$$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{13}}{2} \text{ can also be expressed as}$$

$$x = \frac{3 \pm \sqrt{13}}{2}.$$

Continued on next page



Label key features on the graph, using exact values.

Now you try

Sketch these graphs by completing the square, giving the x -intercepts in exact form.

a $y = x^2 - 2x + 2$

b $y = x^2 + 4x + 1$

c $y = x^2 - 5x + 1$

Exercise 7D

FLUENCY

1, $2 - 6(\frac{1}{2})$

$1 - 6(\frac{1}{2})$

$1 - 6(\frac{1}{3})$

Example 9a

- 1** State whether the turning points of the following are a maximum or a minimum and give the coordinates.

a $y = 2(x - 3)^2 + 5$

b $y = -2(x - 1)^2 + 3$

c $y = -4(x + 1)^2 - 2$

d $y = 6(x + 2)^2 - 5$

e $y = 3(x + 5)^2 + 10$

f $y = -4(x - 7)^2 + 2$

g $y = -5(x - 3)^2 + 8$

h $y = 2(x - 3)^2 - 7$

Example 9b

- 2** Determine the coordinates of the y -intercept of each of the following.

a $y = (x + 1)^2 + 5$

b $y = (x + 2)^2 - 6$

c $y = (x - 3)^2 - 2$

d $y = (x - 4)^2 - 7$

e $y = -(x + 5)^2 + 9$

f $y = -(x - 7)^2 - 6$

g $y = x^2 + 6x + 3$

h $y = x^2 + 5x + 1$

i $y = x^2 + 7x - 5$

j $y = x^2 + x - 8$

k $y = x^2 - 5x + 13$

l $y = x^2 - 12x - 5$

Example 9c

- 3** Determine the coordinates of the x -intercepts (if any) of the following.

a $y = (x - 3)^2 - 4$

b $y = (x + 4)^2 - 9$

c $y = (x - 3)^2 - 36$

d $y = 2(x + 2)^2 - 10$

e $y = -3(x - 1)^2 + 30$

f $y = (x - 5)^2 - 3$

g $y = (x - 4)^2$

h $y = (x + 6)^2$

i $y = 2(x - 7)^2 + 18$

j $y = -2(x - 3)^2 - 4$

k $y = -(x - 2)^2 + 5$

l $y = -(x - 3)^2 + 10$

- 4** Determine the coordinates of the x -intercepts (if any) by first completing the square and rewriting the equation in turning point form. Give exact answers.

a $y = x^2 + 6x + 5$

b $y = x^2 + 6x + 2$

c $y = x^2 + 8x - 5$

d $y = x^2 + 2x - 6$

e $y = x^2 - 4x + 14$

f $y = x^2 - 12x - 5$

- 5 Sketch the graphs of the following. Label the turning point and intercepts.

a $y = (x - 2)^2 - 4$

b $y = (x + 4)^2 - 9$

c $y = (x + 4)^2 - 1$

d $y = (x - 3)^2 - 4$

e $y = (x + 8)^2 + 16$

f $y = (x + 7)^2 + 2$

g $y = (x - 2)^2 + 1$

h $y = (x - 3)^2 + 6$

i $y = -(x - 5)^2 - 4$

j $y = -(x + 4)^2 - 9$

k $y = -(x + 9)^2 + 25$

l $y = -(x - 2)^2 + 4$

Example 10a,b

- 6 Sketch these graphs by completing the square. Label the turning point and intercepts.

a $y = x^2 - 2x + 6$

b $y = x^2 + 4x + 3$

c $y = x^2 - 2x - 3$

d $y = x^2 + 6x + 9$

e $y = x^2 - 8x + 16$

f $y = x^2 - 8x + 20$

g $y = x^2 + 8x + 10$

h $y = x^2 + 6x - 5$

i $y = x^2 + 12x$

PROBLEM-SOLVING

7

7-8($\frac{1}{2}$)

7-9($\frac{1}{2}$)

- 7 Complete the square and decide if the graphs of the following quadratics will have zero, one or two x -intercepts.

a $y = x^2 - 4x + 2$

b $y = x^2 - 4x + 4$

c $y = x^2 + 6x + 9$

d $y = x^2 + 2x + 6$

e $y = x^2 - 6x + 12$

f $y = x^2 + 10x + 20$

Example 10c

- 8 Sketch these graphs by completing the square. Label the turning point and intercepts with exact values.

a $y = x^2 - 3x + 1$

b $y = x^2 + 5x + 2$

c $y = x^2 - x - 2$

d $y = x^2 + 3x + 3$

- 9 Take out a common factor and complete the square to find the y -coordinate of the x -intercepts for these quadratics.

a $y = 2x^2 + 4x - 10$

b $y = 3x^2 - 12x + 9$

c $y = 2x^2 - 12x - 14$

d $y = 4x^2 + 16x - 24$

e $y = 5x^2 + 20x - 35$

f $y = 2x^2 - 6x + 2$

REASONING

10($\frac{1}{2}$)

10($\frac{1}{2}$), 11

10($\frac{1}{3}$), 11, 12($\frac{1}{3}$), 13

- 10 To sketch a graph of the form $y = -x^2 + bx + c$ we can complete the square by taking out a factor of -1 . Here is an example.

$$\begin{aligned} y &= -x^2 - 2x + 5 \\ &= -(x^2 + 2x - 5) \\ &= -\left(x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 - 5\right) \\ &= -((x + 1)^2 - 6) \\ &= -(x + 1)^2 + 6 \end{aligned}$$

So the turning point is a maximum at $(-1, 6)$.

Sketch the graph of these quadratics using the technique above.

a $y = -x^2 - 4x + 3$

b $y = -x^2 + 2x + 2$

c $y = -x^2 + 6x - 4$

d $y = -x^2 + 8x - 8$

e $y = -x^2 - 3x - 5$

f $y = -x^2 - 5x + 2$

- 11 For what values of k will the graph of $y = (x - h)^2 + k$ have:
a zero x -intercepts? **b** one x -intercept? **c** two x -intercepts?

- 12 This example recalls how to complete the square with non-monic quadratics of the form $y = ax^2 + bx + c$.

$$\begin{aligned} y &= 3x^2 + 6x + 1 \\ &= 3\left(x^2 + 2x + \frac{1}{3}\right) \\ &= 3\left(x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + \frac{1}{3}\right) \\ &= 3\left((x + 1)^2 - \frac{2}{3}\right) \\ &= 3(x + 1)^2 - 2 \end{aligned}$$

Key features:

The turning point is $(-1, -2)$.

y -intercept is at $(0, 1)$.

x -intercepts:

$$0 = 3(x + 1)^2 - 2$$

$$(x + 1)^2 = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}} - 1$$

Use this technique to sketch the graphs of these non-monic quadratics.

- | | |
|-------------------------------|--------------------------------|
| a $y = 4x^2 + 8x + 3$ | b $y = 3x^2 - 12x + 10$ |
| c $y = 2x^2 + 12x + 1$ | d $y = 2x^2 + x - 3$ |
| e $y = 2x^2 - 7x + 3$ | f $y = 4x^2 - 8x + 20$ |
| g $y = 6x^2 + 5x + 9$ | h $y = 5x^2 - 3x + 7$ |
| i $y = 5x^2 + 12x$ | j $y = 7x^2 + 10x$ |
| k $y = -3x^2 - 9x + 2$ | l $y = -4x^2 + 10x - 1$ |

- 13 Show that $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}$.

ENRICHMENT: Finding rules using the turning point

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14

- 14 Find the rule of the quadratic graph with the following features. Express in the form $y = a(x - h)^2 + k$.
- a** turning point at $(3, 2)$ and passes through $(0, 20)$
 - b** turning point at $(-2, 4)$ and passes through $(0, 6)$
 - c** turning point at $(-1, 2)$ and passes through $(-2, 0)$
 - d** turning point at $(2, 0)$ and passes through $(8, 12)$
 - e** axis of symmetry at $x = 2$ and passes through $(0, 0)$ and $(3, -9)$
 - f** axis of symmetry at $x = -1$ and passes through $(0, 4)$ and $(2, -4)$

7E Sketching parabolas using the quadratic formula and the discriminant

Learning intentions for this section:

- To know the quadratic formula and use it to find the solutions of a quadratic equation
- To be able to use the quadratic formula to determine the x -intercepts of a quadratic graph
- To understand how the discriminant can be used to determine the number of x -intercepts
- To be able to use the axis of symmetry rule to locate the turning point of a quadratic graph

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

So far we have found x -intercepts for parabolas by factorising (and using the Null Factor Law) and by completing the square. An alternative method is to use the quadratic formula, which states that if

$$ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

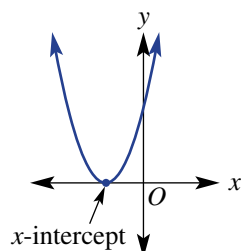
The discriminant $\Delta = b^2 - 4ac$ determines the number of solutions to the equation.

If $\Delta = 0$, i.e. $b^2 - 4ac = 0$.

The solution to the equation

$$\text{becomes } x = -\frac{b}{2a}.$$

There is one solution and one x -intercept.

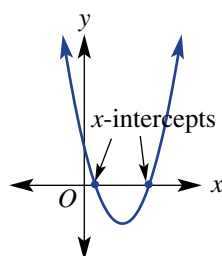


If $\Delta > 0$, i.e. $b^2 - 4ac > 0$.

The solution to the equation

$$\text{becomes } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

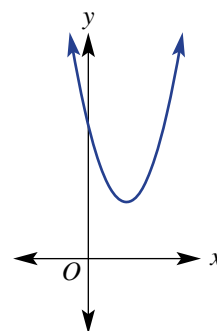
There are two solutions and two x -intercepts.



If $\Delta < 0$, i.e. $b^2 - 4ac < 0$.

Square roots exist for positive numbers only.

There are no solutions nor x -intercepts.



Lesson starter: No working required

Three students set to work to find the x -intercepts for $y = x^2 - 2x + 3$:

Student A finds the intercepts by factorising.

Student B finds the intercepts by completing the square.

Student C uses the discriminant in the quadratic formula.

- Try the method for student A. What do you notice?
- Try the method for student B. What do you notice?

- What is the value of the discriminant for student C? What does this tell them about the number of x -intercepts for the quadratic?
- What advice would student C give students A and B?

KEY IDEAS

■ To sketch the graph of $y = ax^2 + bx + c$, find the following points.

- y -intercept at $x = 0$: $y = a(0)^2 + b(0) + c = c$
- x -intercepts when $y = 0$:

For $0 = ax^2 + bx + c$, use the **quadratic formula**:

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Alternatively, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Turning point: The x -coordinate lies halfway between the x -coordinates of the x -intercepts, so $x = -\frac{b}{2a}$.

The y -coordinate is found by substituting the x -coordinate into the original equation.

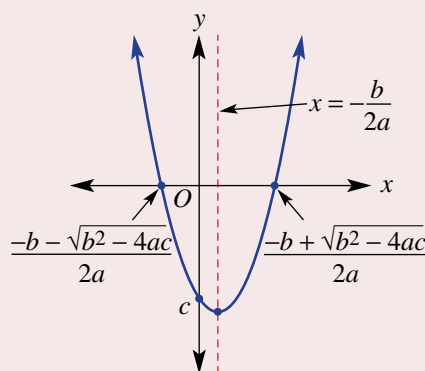
■ $x = -\frac{b}{2a}$ is the **axis of symmetry**.

■ To determine if there are zero, one or two x -intercepts, use the **discriminant** $\Delta = b^2 - 4ac$.

If $\Delta < 0 \rightarrow$ no x -intercepts.

If $\Delta = 0 \rightarrow$ one x -intercept.

If $\Delta > 0 \rightarrow$ two x -intercepts.



BUILDING UNDERSTANDING

1 A graph has the rule $y = ax^2 + bx + c$. Determine the number of x -intercepts it will have if:

a $b^2 - 4ac > 0$

b $b^2 - 4ac < 0$

c $b^2 - 4ac = 0$

2 Give the exact value of $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when:

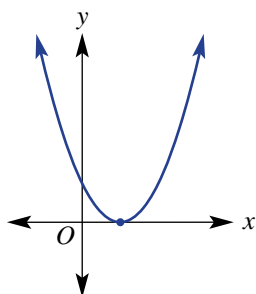
a $a = 1, b = 2, c = -1$

b $a = -2, b = 3, c = 5$

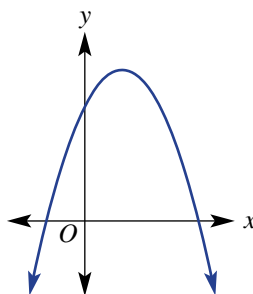
c $a = -2, b = -1, c = 2$

3 For the following graphs, state whether the discriminant of these quadratics would be zero, positive or negative.

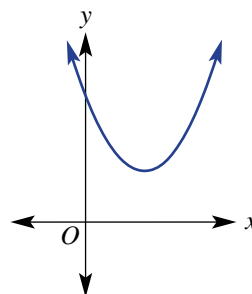
a



b



c





Example 11 Using the discriminant and using $x = -\frac{b}{2a}$ to find the turning point

Consider the parabola given by the quadratic equation $y = 3x^2 - 6x + 5$.

- a Determine the number of x -intercepts.
- b Determine the coordinates of the y -intercept.
- c Use $x = -\frac{b}{2a}$ to determine the turning point.

SOLUTION

- a $\Delta = b^2 - 4ac$
 $= (-6)^2 - 4(3)(5)$
 $= -24$
 $\Delta < 0$, so there are no x -intercepts.
- b y -intercept is at $(0, 5)$.
- c $x = -\frac{b}{2a}$
 $= -\frac{(-6)}{2(3)}$
 $= 1$
 $y = 3(1)^2 - 6(1) + 5$
 $= 2$
 $\therefore$ Turning point is at $(1, 2)$.

EXPLANATION

Use the discriminant $\Delta = b^2 - 4ac$ to find the number of x -intercepts. In $3x^2 - 6x + 5$, $a = 3$, $b = -6$ and $c = 5$. Interpret the result.

Substitute $x = 0$ for the y -intercept.

For the x -coordinate of the turning point use $x = -\frac{b}{2a}$ with $a = 3$ and $b = -6$, as above.

Substitute the x -coordinate into $y = 3x^2 - 6x + 5$ to find the corresponding y -coordinate of the turning point.

Now you try

Consider the parabola given by the quadratic equation $y = 2x^2 - 4x + 1$.

- a Determine the number of x -intercepts.
- b Determine the coordinates of the y -intercept.
- c Use $x = -\frac{b}{2a}$ to determine the turning point.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Example 12 Sketching graphs using the quadratic formula

Sketch the graph of the quadratic $y = 2x^2 + 4x - 3$, labelling all significant points. Round the x -intercepts to two decimal places.

SOLUTION

y -intercept is at $(0, -3)$.

x -intercepts ($y = 0$) :

$$2x^2 + 4x - 3 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{40}}{4} \end{aligned}$$

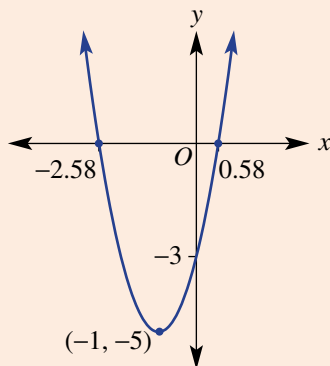
$$x = 0.58, -2.58 \text{ (to 2 d.p.)}$$

$\therefore x$ -intercepts are at $(0.58, 0)$, $(-2.58, 0)$

$$\begin{aligned} \text{Turning point is at } x &= -\frac{b}{2a} \\ &= -\frac{(4)}{2(2)} \\ &= -1 \end{aligned}$$

$$\text{and } \therefore y = 2(-1)^2 + 4(-1) - 3 = -5.$$

$\therefore$ Turning point is at $(-1, -5)$.



EXPLANATION

Identify key features; i.e. x - and y -intercepts and the turning point. Substitute $x = 0$ for the y -intercept.

Use the quadratic formula to find the x -intercepts.

For $y = 2x^2 + 4x - 3$, $a = 2$, $b = 4$ and $c = -3$.

$$\begin{aligned} \text{Note: } \frac{-4 \pm \sqrt{40}}{4} &= \frac{-4 \pm 2\sqrt{10}}{4} \\ &= \frac{-2 \pm \sqrt{10}}{2} \end{aligned}$$

Use a calculator to round to two decimal places.

Substitute $x = -1$ into $y = 2x^2 + 4x - 3$ to find the y -coordinate of the turning point.

Label the key features on the graph and sketch.

Now you try

Sketch the graph of the quadratic $y = 3x^2 - 6x + 1$, labelling all significant points. Round the x -intercepts to two decimal places.

Exercise 7E

FLUENCY

1-4($\frac{1}{2}$)1-4($\frac{1}{2}$)1-4($\frac{1}{3}$)

Example 11a

- 1 Use the discriminant to determine the number of x -intercepts for the parabolas given by the following quadratics.

a $y = x^2 + 4x + 4$

b $y = x^2 - 3x + 5$

c $y = -x^2 + 4x + 2$

d $y = 3x^2 - 4x - 2$

e $y = 2x^2 - x + 2$

f $y = 2x^2 - 12x + 18$

g $y = 3x^2 - 2x$

h $y = 3x^2 + 5x$

i $y = -3x^2 - 2x$

j $y = 3x^2 + 5$

k $y = 4x^2 - 2$

l $y = -5x^2 + x$

Example 11b

- 2 Determine the coordinates of the y -intercept for the parabolas given by the following quadratics.

a $y = x^2 + 2x + 3$

b $y = x^2 - 4x + 5$

c $y = 4x^2 + 3x - 2$

d $y = 5x^2 - 2x - 4$

e $y = -2x^2 - 5x + 8$

f $y = -2x^2 + 7x - 10$

g $y = 3x^2 + 8x$

h $y = -4x^2 - 3x$

i $y = 5x^2 - 7$

Example 11c

- 3 Use $x = -\frac{b}{2a}$ to determine the coordinates of the turning point for the parabolas defined by the following quadratics.

a $y = x^2 + 2x + 4$

b $y = x^2 + 4x - 1$

c $y = x^2 - 4x + 3$

d $y = -x^2 + 2x - 6$

e $y = -x^2 - 3x + 4$

f $y = -x^2 + 7x - 7$

g $y = 2x^2 + 3x - 4$

h $y = 4x^2 - 3x$

i $y = -4x^2 - 9$

j $y = -4x^2 + 2x - 3$

k $y = -3x^2 - 2x$

l $y = -5x^2 + 2$

Example 12



- 4 Sketch the graph of these quadratics, labelling all significant points. Round the x -intercepts to two decimal places.

a $y = 2x^2 + 8x - 5$

b $y = 3x^2 + 6x - 2$

c $y = 4x^2 - 2x - 3$

d $y = 2x^2 - 4x - 9$

e $y = 2x^2 - 8x - 11$

f $y = 3x^2 + 9x - 10$

g $y = -3x^2 + 6x + 8$

h $y = -2x^2 - 4x + 7$

i $y = -4x^2 + 8x + 3$

j $y = -2x^2 - x + 12$

k $y = -3x^2 - 2x$

l $y = -5x^2 - 10x - 4$

PROBLEM-SOLVING

5($\frac{1}{2}$)5-6($\frac{1}{2}$)5-6($\frac{1}{3}$), 7

- 5 Give the exact x -intercepts of the graphs of these parabolas. Simplify any surds.

a $y = 3x^2 - 6x - 1$

b $y = -2x^2 - 4x + 3$

c $y = -4x^2 + 8x + 6$

d $y = 2x^2 + 6x - 3$

e $y = 2x^2 - 8x + 5$

f $y = 5x^2 - 10x - 1$

- 6 Sketch the graphs of these quadratics.

a $y = 4x^2 + 12x + 9$

b $y = 9x^2 - 6x + 1$

c $y = -4x^2 - 20x - 25$

d $y = -9x^2 + 30x - 25$

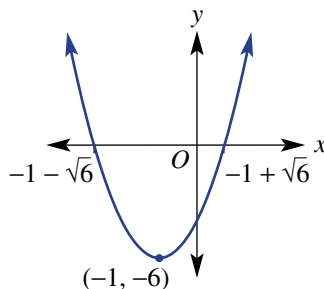
e $y = -2x^2 + 8x - 11$

f $y = -3x^2 + 12x - 16$

g $y = 4x^2 + 4x + 3$

h $y = 3x^2 + 4x + 2$

- 7 Find a rule in the form $y = ax^2 + bx + c$ that matches this graph.

**REASONING**

8

8, 9

9, 10

- 8 Write down two rules in the form $y = ax^2 + bx + c$ that have:
a two x -intercepts **b** one x -intercept **c** no x -intercepts.
- 9 Explain why the quadratic formula gives only one solution when the discriminant ($b^2 - 4ac$) is equal to 0.
- 10 Write down the quadratic formula for monic quadratic equations (i.e. where $a = 1$).

ENRICHMENT: Some proof

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11, 12

- 11 Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$ to find the general rule for the y -coordinate of the turning point in terms of a , b and c .
- 12 Prove the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by solving $ax^2 + bx + c = 0$.
 (Hint: Divide both sides by a and complete the square.)

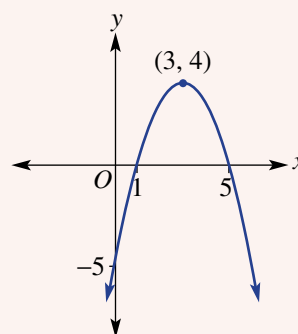


One of the most important applications of quadratic equations is in modelling acceleration, first formulated by Galileo in the early 1600s, and shown in action here in the present day.

7A

1 For the given graph state:

- a the turning point and whether it is a maximum or minimum
- b the axis of symmetry
- c the coordinates of the intercepts.



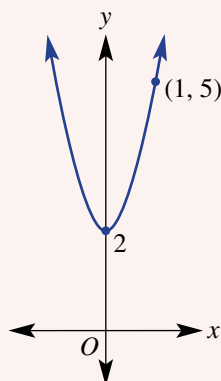
7B

2 Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept. (The x-intercepts are *not* required.)

- a $y = 2x^2$ b $y = -x^2 + 3$ c $y = (x - 3)^2$ d $y = -(x + 2)^2 - 1$

7B

3 Find a rule for this parabola with turning point (0, 2) and another point (1, 5).

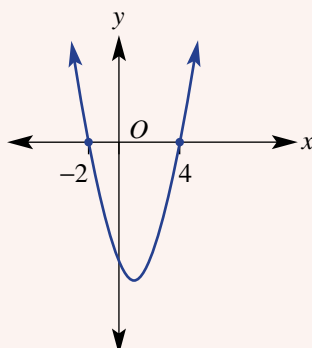


7C

4 Sketch graphs of the following quadratics, label the x- and y-intercepts and determine the coordinates of the turning point, using symmetry.

- a $y = x^2 - 2x - 3$ b $y = x^2 - 4x + 4$

7C

5 The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b , then find the coordinates of the turning point.

7D

6 For $y = -2(x - 3)^2 + 8$ determine the:

- a coordinates of the turning point and state whether it is a maximum or minimum
- b y-intercept coordinates
- c x-intercept coordinates (if any).

7D

7 Sketch these graphs by first completing the square to write the equation in turning point form. Label the exact x- and y-intercepts and turning point on the graph.

a $y = x^2 - 4x + 3$

b $y = x^2 - 2x - 6$

7E

8 For the parabolas given by the following quadratic equations:

- i use the discriminant to determine the number of x-intercepts
- ii determine the coordinates of the y-intercept
- iii use $x = -\frac{b}{2a}$ to determine the turning point.

a $y = x^2 - 4x + 5$

b $y = x^2 + 6x - 7$

c $y = -x^2 - 8x - 16$

7E

9 Sketch the graph of the quadratic $y = 2x^2 - 8x + 5$, labelling all significant points. Give the x-intercepts, rounded to two decimal places.

7F Applications of parabolas

Learning intentions for this section:

- To be able to set up a quadratic model to solve a word problem
- To know how to apply the processes of quadratics to identify key features of a graph and relate them to real-life contexts
- To be able to identify the possible values of a variable in a given context

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- It consolidates and extends the concepts in Chapter 10 of our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

Quadratic equations and their graphs can be used to solve a range of practical problems. These could involve, for example, the path of a projectile or the shape of a bridge's arch. We can relate quantities with quadratic rules and use their graphs to illustrate key features. For example, x -intercepts show where one quantity (y) is equal to zero, and the turning point is where a quantity is a maximum or minimum.

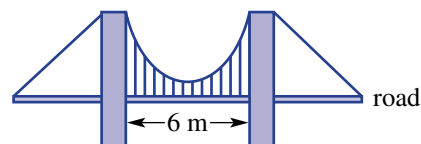


Engineers could design a suspension bridge by placing the road on the x -axis and a pylon on the y -axis. The support cable forms a parabola and its quadratic equation is used to find the heights of the evenly spaced, vertical supports.

Lesson starter: The civil engineer

Michael, a civil engineer, designs a model for the curved cable of a 6 m suspension bridge using the equation $h = (d - 3)^2 + 2$, where h metres is the height of the hanging cables above the road for a distance d metres from the left pillar.

- What are the possible values for d ?
- Sketch the graph of $h = (d - 3)^2 + 2$ for appropriate values of d .
- What is the height of the pillars above the road?
- What is the minimum height of the cable above the road?
- Discuss how key features of the graph have helped to answer the questions above.

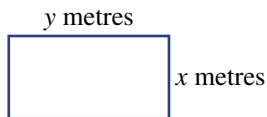
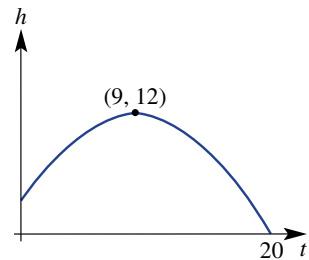


KEY IDEAS

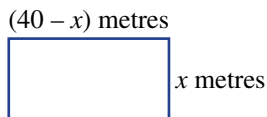
- Applying quadratics to solve problems may involve:
 - defining variables
 - forming equations
 - solving equations
 - deciding on a suitable range of values for the variables
 - sketching graphs showing key features
 - finding the maximum or minimum turning point.

BUILDING UNDERSTANDING

- 1 This graph shows the height (h metres) of a stone thrown through the air after t seconds.
- How many seconds does it take for the stone to reach its maximum height?
 - What is the maximum height of the stone?
 - What horizontal distance does the stone travel?
- 2 a If the perimeter of this rectangle is 20 metres, write an equation in terms of x and y .



- b Give an expression for the area of this rectangle.



 Example 13 Applying parabolas given the rule

The path of a javelin thrown by Jo is given by the formula $h = -\frac{1}{16}(d - 10)^2 + 9$, where h metres is the height of the javelin above the ground and d metres is the horizontal distance travelled.

- Sketch the graph of the rule for $0 \leq d \leq 22$ by finding the intercepts and the coordinates of the turning point.
- What is the maximum height the javelin reaches?
- What horizontal distance does the javelin travel (i.e. when is $h = 0$)?

SOLUTION

- a Turning point is $(10, 9)$

h -intercept ($d=0$):

$$h = -\frac{1}{16}(-10)^2 + 9 = 2.75$$

d -intercepts ($h=0$)

$$0 = -\frac{1}{16}(d - 10)^2 + 9$$

$$144 = (d - 10)^2$$

$$d - 10 = +/\!-\sqrt{144}$$

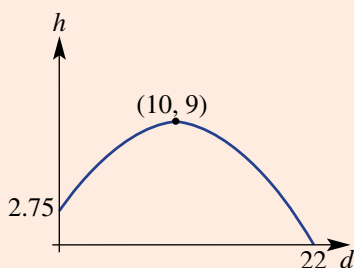
$$d = -2 \text{ or } 22$$

EXPLANATION

If $y = a(x - h)^2 + k$, then the turning point is (h, k) .

Substitute $d = 0$ to find the h -intercept.

Substitute $h = 0$ and solve to find the d -intercepts.



- b** 9 metres
c 22 metres

The maximum value of h is at the turning point.
 $d = 22$ when $h = 0$.

Now you try

A ball is thrown upwards from ground level and reaches a height of h metres after t seconds, given by the formula $h = 20t - 5t^2$.

- a** Sketch a graph of the rule for $0 \leq t \leq 4$ by finding the t -intercepts (x -intercepts) and the coordinates of the turning point.
b What maximum height does the ball reach?
c How long does it take the ball to return to ground level ($h = 0$)?



Example 14 Applying parabolas by formulating a rule

A piece of wire measuring 100 cm in length is bent into the shape of a rectangle. Let x cm be the breadth of the rectangle.

- a** Use the perimeter to write an expression for the length of the rectangle in terms of x .
b Write an equation for the area of the rectangle ($A \text{ cm}^2$) in terms of x .
c Decide on the suitable values of x .
d Sketch the graph of A versus x for suitable values of x .
e Use the graph to determine the maximum area that can be formed.
f What will be the dimensions of the rectangle to achieve its maximum area?

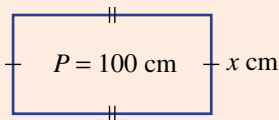
SOLUTION

- a** $2 \times \text{length} + 2x = 100$
 $2 \times \text{length} = 100 - 2x$
 $\therefore \text{Length} = 50 - x$

b $A = x(50 - x)$

- c** Length and breadth must be positive, so we require:
 $x > 0$ and $50 - x > 0$
 i.e. $x > 0$ and $50 > x$
 i.e. $0 < x < 50$

EXPLANATION

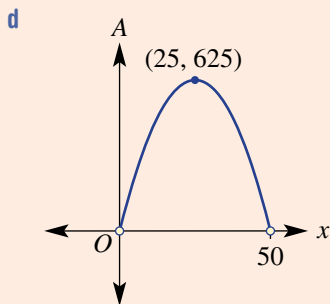


100 cm of wire will form the perimeter.
 Length is half of $(100 - 2 \times \text{breadth})$.

Area of a rectangle = length $\times$ breadth.

Require each dimension to be positive, solve for x .

Continued on next page



e The maximum area that can be formed is 625 cm^2 .

f Maximum occurs when breadth $x = 25 \text{ cm}$, so
 Length $= 50 - 25$
 $= 25 \text{ cm}$
 Dimensions that give maximum area are
 25 cm by 25 cm, which is, in fact, a square.

Sketch the graph, labelling the intercepts and turning point, which has x -coordinate halfway between the x -intercepts; i.e. $x = 25$. Substitute $x = 25$ into the area formula to find the maximum area: $A = 25(50 - 25) = 625$. Note open circles at $x = 0$ and $x = 50$ as these points are not included in the possible x -values.

Read from the graph. The maximum area is the y -coordinate of the turning point.

From turning point, $x = 25$ gives the maximum area. Substitute to find the corresponding length. Length $= 50 - x$.

Now you try

A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let $x \text{ cm}$ be the breadth of the rectangle.

- Use the perimeter to write an expression for the length of the rectangle in terms of x .
- Write an equation for the area of the rectangle ($A \text{ cm}^2$) in terms of x .
- Decide on the suitable values of x .
- Sketch the graph of A versus x for suitable values of x .
- Use the graph to determine the maximum area that can be formed.
- What will be the dimensions of the rectangle to achieve its maximum area?

Exercise 7F

FLUENCY

1–3

1–4

2–5

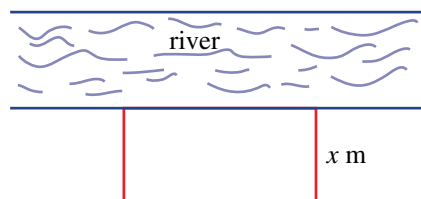
Example 13

- A wood turner carves out a bowl according to the formula $d = \frac{1}{3}x^2 - 27$, where $d \text{ cm}$ is the depth of the bowl and $x \text{ cm}$ is the distance from the centre of the bowl.
 - Sketch a graph for $-9 \leq x \leq 9$, showing x -intercepts and the turning point.
 - What is the breadth of the bowl?
 - What is the maximum depth of the bowl?
- The equation for the arch of a particular bridge is given by $h = -\frac{1}{500}(x - 100)^2 + 20$, where $h \text{ m}$ is the height above the base of the bridge and $x \text{ m}$ is the distance from the left side.
 - Determine the coordinates of the turning point of the graph.
 - Determine the x -intercepts of the graph.
 - Sketch the graph of the arch for appropriate values of x .
 - What is the span of the arch?
 - What is the maximum height of the arch?

Example 14

- 3 A 20 cm piece of wire is bent to form a rectangle. Let x cm be the breadth of the rectangle.
- Use the perimeter to write an expression for the length of the rectangle in terms of x .
 - Write an equation for the area of the rectangle (A cm²) in terms of x .
 - Decide on suitable values of x .
 - Sketch the graph of A versus x for suitable values of x .
 - Use the graph to determine the maximum area that can be formed.
 - What will be the dimensions of the rectangle to achieve its maximum area?

- 4 A farmer has 100 m of fencing to form a rectangular paddock with a river on one side (that does not require fencing), as shown.



- Use the perimeter to write an expression for the length of the paddock in terms of the breadth, x metres.
 - Write an equation for the area of the paddock (A m²) in terms of x .
 - Decide on suitable values of x .
 - Sketch the graph of A versus x for suitable values of x .
 - Use the graph to determine the maximum paddock area that can be formed.
 - What will be the dimensions of the paddock to achieve its maximum area?
- 5 The sum of two positive numbers is 20 and x is the smaller number.
- Write the second number in terms of x .
 - Write a rule for the product, P , of the two numbers in terms of x .
 - Sketch a graph of P vs x .
 - Find the values of x when:
 - $P = 0$
 - P is a maximum.
 - What is the maximum value of P ?

PROBLEM-SOLVING

6

6, 7

7, 8

- 6 The equation for a support span is given by $h = -\frac{1}{40}(x - 20)^2$, where h m is the distance below the deck of a bridge and x m is the distance from the left side.
- Determine the coordinates of the turning point of the graph.
 - Sketch a graph of the equation using $0 \leq x \leq 40$.
 - What is the breadth of the support span?
 - What is the maximum height of the support span?
- 7 Jordie throws a rock from the top of a 30 metre high cliff and its height (h metres) above the sea is given by $h = 30 - 5t^2$, where t is in seconds.
- Find the exact time it takes for the rock to hit the water.
 - Sketch a graph of h vs t for appropriate values of t .
 - What is the exact time it takes for the rock to fall to a height of 20 metres?

- 8 A bird dives into the water to catch a fish. It follows a path given by $h = t^2 - 8t + 7$, where h is the height in metres above sea level and t is the time in seconds.
- Sketch a graph of h vs t , showing intercepts and the turning point.
 - Find the time when the bird:
 - enters the water
 - exits the water
 - reaches a maximum depth.
 - What is the maximum depth to which the bird dives?
 - At what times is the bird at a depth of 8 metres?

REASONING

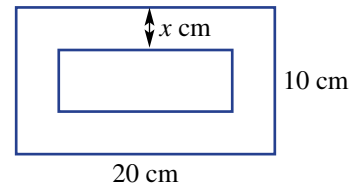
9

9–11

11, 12

- 9 The height, h metres, of a flying kite is given by the rule $h = t^2 - 6t + 10$ for t seconds.
- Find the minimum height of the kite during this time.
 - Does the kite ever hit the ground during this time? Give reasons.
- 10 The sum of two numbers is 64. Show that their product has a maximum of 1024.

- 11 A rectangular framed picture has a total length and breadth of 20 cm and 10 cm, respectively. The frame has breadth x cm.



- Find the rule for the area ($A \text{ cm}^2$) of the picture inside.
 - What are the minimum and maximum values of x ?
 - Sketch a graph of A vs x using suitable values of x .
 - Explain why there is no turning point for your graph, using suitable values of x .
 - Find the breadth of the frame if the area of the picture is 144 cm^2 .
- 12 A dolphin jumping out of the water follows a path described by $h = -\frac{1}{2}(x^2 - 10x + 16)$, where h is the vertical height, in metres, and x metres is the horizontal distance travelled.
- How far horizontally does the dolphin travel out of the water?
 - Does the dolphin ever reach a height of 5 metres above water level? Give reasons.

ENRICHMENT: The highway and the river and the lobbed ball

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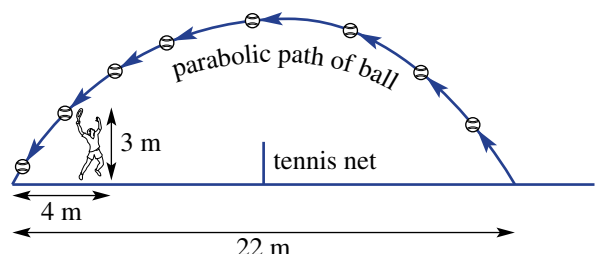
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13, 14

- 13 The path of a river is given by the rule $y = \frac{1}{10}x(x - 100)$ and all units are given in metres. A highway is to be built near or over the river on the line $y = c$.
- Sketch a graph of the path of the river, showing key features.
 - For the highway with equation $y = c$, decide how many bridges will need to be built if:
 - $c = 0$
 - $c = -300$
 - Locate the coordinates of the bridge, correct to one decimal place, if:
 - $c = -200$
 - $c = -10$
 - Describe the situation when $c = -250$.



- 14 A tennis ball is lobbed from ground level and must cover a horizontal distance of 22 m if it is to land just inside the opposite end of the court. If the opponent is standing 4 m from the baseline and he can hit any ball less than 3 m high, what is the lowest maximum height the lob must reach to win the point?



7G Intersection of lines and parabolas

Learning intentions for this section:

- To understand how a line can intersect a parabola at 0, 1 or 2 points
- To be able to find the points of intersection of a line and a parabola using substitution
- To know that the discriminant can be used to determine the number of points of intersection of a line and a parabola

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Non-linear relationships C*
- These concepts were not addressed in our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

We have seen previously when simultaneously solving a pair of linear equations that there is one solution provided that the graphs of these linear equations are not parallel. Graphically, this represents the point of intersection for the two straight lines.

For the intersection of a parabola and a line we can have either zero, one or two points of intersection. As we have done for linear simultaneous equations, we can use the method of substitution to solve a linear equation and a non-linear equation simultaneously.



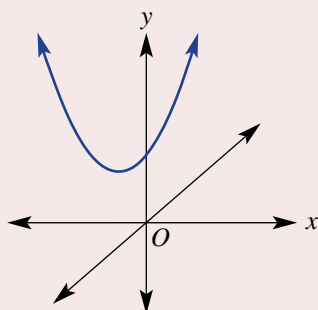
This image shows why the Infinity Bridge, England, is named after the symbol ∞ . Architects develop and solve equations in three dimensions to determine the intersection points of lines (support cables) and parabolic curves (the arches).

Lesson starter: How many times does a line cut a parabola?

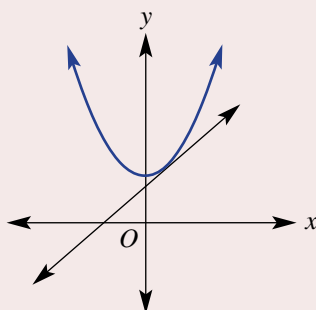
- Use computer graphing software to plot a graph of $y = x^2$.
- By plotting lines of the form $x = h$, determine how many points of intersection a vertical line will have with the parabola.
- By plotting lines of the form $y = c$, determine how many points of intersection a horizontal line could have with the parabola.
- By plotting straight lines of the form $y = 2x + k$ for various values of k , determine the number of possible intersections between a line and a parabola.
- State some values of k for which the line above intersects the parabola:
 - twice
 - never.
- Can you find the value of k for which the line intersects the parabola exactly once?

KEY IDEAS

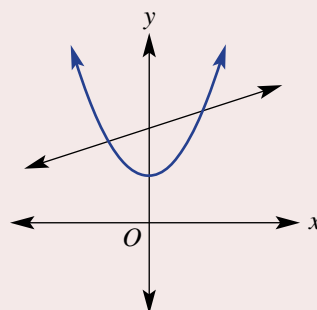
- When solving a pair of simultaneous equations involving a parabola and a line, we may obtain zero, one or two solutions. Graphically, this represents zero, one or two points of intersection between the parabola and the line.
 - A line that intersects a curve twice is called a **secant**.
 - A line that touches a curve at a single point of contact is called a **tangent**.



zero points of intersection



one point of intersection



two points of intersection

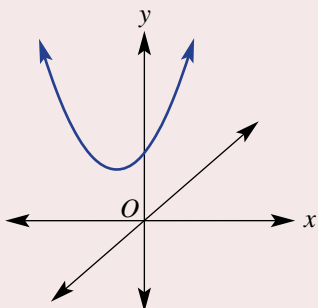
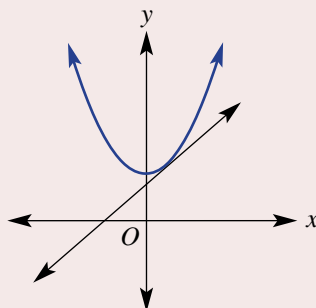
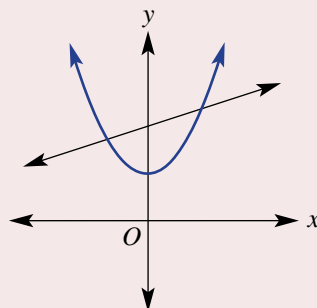
■ The method of substitution is used to solve the equations simultaneously.

- Substitute one equation into the other.
- Rearrange the resulting equation into the form $ax^2 + bx + c = 0$.
- Solve for x by factorising and applying the Null Factor Law or use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Substitute the x -values into one of the original equations to find the corresponding y -value.

■ After substituting the equations and rearranging, we arrive at an equation of the form $ax^2 + bx + c = 0$. Hence, the discriminant, $b^2 - 4ac$, can be used to determine the number of solutions (i.e. points of intersection) of the two equations.

zero solutions
 $b^2 - 4ac < 0$ one solution
 $b^2 - 4ac = 0$ two solutions
 $b^2 - 4ac > 0$

BUILDING UNDERSTANDING

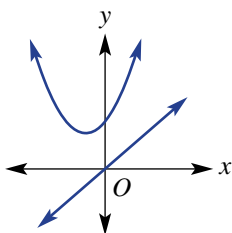
- a** Find the coordinates of the point where the vertical line $x = 2$ intersects the parabola $y = 2x^2 + 5x - 6$.

b Find the coordinates of the point where the vertical line $x = -1$ intersects the parabola $y = x^2 + 3x - 1$.
- Rearrange the following into the form $ax^2 + bx + c = 0$, where $a > 0$.

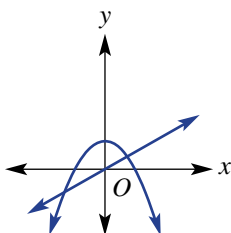
a $x^2 + 5x = 2x - 6$ **b** $x^2 - 3x + 4 = 2x + 1$ **c** $x^2 + x - 7 = -2x + 5$

- 3 What do we know about the discriminant, $b^2 - 4ac$, of the resulting equation from solving the equations which correspond to these graphs simultaneously?

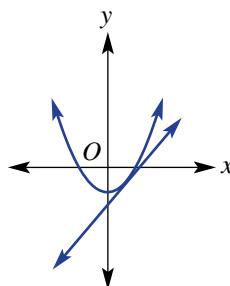
a



b



c



Example 15 Finding points of intersection of a parabola and a horizontal line

Find any points of intersection of these parabolas and horizontal lines.

a $y = x^2 - 3x$
 $y = 4$

b $y = x^2 + 2x + 4$
 $y = -2$

SOLUTION

a By substitution:

$$\begin{aligned}x^2 - 3x &= 4 \\x^2 - 3x - 4 &= 0 \\(x - 4)(x + 1) &= 0 \\x - 4 = 0 \text{ or } x + 1 &= 0 \\x = 4 \text{ or } x &= -1\end{aligned}$$

$\therefore$ The points of intersection are at (4, 4) and (-1, 4).

b By substitution:

$$\begin{aligned}x^2 + 2x + 4 &= -2 \\x^2 + 2x + 6 &= 0 \\x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\x &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)} \\&= \frac{-2 \pm \sqrt{-20}}{2}\end{aligned}$$

$\therefore$ There are no points of intersection.

EXPLANATION

Substitute $y = 4$ from the second equation into the first equation.

Write in the form $ax^2 + bx + c = 0$ by subtracting 4 from both sides.

Factorise and apply the Null Factor Law to solve for x . As the points are on the line $y = 4$, the y -coordinate of the points of intersection is 4.

Substitute $y = -2$ into the first equation.

Apply the quadratic formula to solve $x^2 + 2x + 6 = 0$, where $a = 1$, $b = 2$ and $c = 6$. Alternatively, complete the square: $(x + 1)^2 + 5 = 0$.

$\sqrt{-20}$ has no real solutions, or using $(x + 1)^2 + 5 = 0$, there are no solutions.

The parabola $y = x^2 + 2x + 4$ and the line $y = -2$ do not intersect.

Now you try

Find any points of intersection of these parabolas and horizontal lines.

a $y = x^2 - x$
 $y = 2$

b $y = x^2 + 4x + 1$
 $y = -4$



Example 16 Solving simultaneous equations involving a line and a parabola

Solve the following equations simultaneously.

a $y = x^2$
 $y = 2x$

b $y = -4x^2 - x + 6$
 $y = 3x + 7$

c $y = x^2 + 1$
 $2x - 3y = -4$

SOLUTION

a By substitution:

$$\begin{aligned}x^2 &= 2x \\x^2 - 2x &= 0 \\x(x - 2) &= 0 \\x = 0 \text{ or } x - 2 &= 0 \\x = 0 \text{ or } x &= 2\end{aligned}$$

When $x = 0$, $y = 2 \times (0) = 0$.

When $x = 2$, $y = 2 \times (2) = 4$.

$\therefore$ The solutions are $x = 0$, $y = 0$ and
 $x = 2$, $y = 4$.

b By substitution:

$$\begin{aligned}-4x^2 - x + 6 &= 3x + 7 \\-x + 6 &= 4x^2 + 3x + 7 \\6 &= 4x^2 + 4x + 7 \\0 &= 4x^2 + 4x + 1 \\\therefore (2x + 1)(2x + 1) &= 0 \\2x + 1 &= 0 \\x &= -\frac{1}{2}\end{aligned}$$

When $x = -\frac{1}{2}$, $y = 3 \times \left(-\frac{1}{2}\right) + 7$
 $= \frac{11}{2}$ or $5\frac{1}{2}$

$\therefore$ The only solution is $x = -\frac{1}{2}$,
 $y = \frac{11}{2}$.

c By substitution:

$$\begin{aligned}2x - 3(x^2 + 1) &= -4 \\2x - 3x^2 - 3 &= -4 \\2x - 3 &= 3x^2 - 4 \\2x &= 3x^2 - 1\end{aligned}$$

$$\begin{aligned}\therefore 3x^2 - 2x - 1 &= 0 \\(3x + 1)(x - 1) &= 0 \\3x + 1 = 0 \text{ or } x - 1 &= 0 \\x = -\frac{1}{3} \text{ or } x &= 1\end{aligned}$$

EXPLANATION

Substitute $y = 2x$ into $y = x^2$.

Rearrange the equation so that it is equal to 0.

Factorise by removing the common factor x .

Apply the Null Factor Law to solve for x .

Substitute the x -values into $y = 2x$ to obtain the corresponding y -value. Alternatively, the equation $y = x^2$ can be used to find the y -values or it can be used to check the y -values.

The points $(0, 0)$ and $(2, 4)$ lie on both the line $y = 2x$ and the parabola $y = x^2$.

Substitute $y = 3x + 7$ into the first equation.

When rearranging the equation equal to 0, gather the terms on the side that makes the coefficient of x^2 positive, as this will make the factorising easier. Hence, add $4x^2$ to both sides, then add x to both sides and subtract 6 from both sides. Factorise and solve for x .

Substitute the x -value into $y = 3x + 7$
(or $y = -4x^2 - x + 6$ but $y = 3x + 7$ is a simpler equation).

Finding only one solution indicates that this line is a tangent to the parabola.

Replace y in $2x - 3y = -4$ with $x^2 + 1$, making sure you include brackets.

Expand the brackets and then rearrange into the form $ax^2 + bx + c = 0$.

Factorise and solve for x .

$$\begin{aligned}\text{When } x = -\frac{1}{3}, y &= \left(-\frac{1}{3}\right)^2 + 1 \\ &= \frac{1}{9} + 1 \\ &= \frac{10}{9}\end{aligned}$$

$$\begin{aligned}\text{When } x = 1, y &= (1)^2 + 1 \\ &= 2\end{aligned}$$

$$\therefore \text{The solutions are } x = -\frac{1}{3}, y = \frac{10}{9} \text{ and } x = 1, y = 2.$$

Substitute the x -values into one of the two original equations to solve for y .

The line and parabola intersect in two places.

Now you try

Solve the following equations simultaneously.

a $y = x^2$
 $y = 4x$

b $y = -x^2 + 6x + 7$
 $y = 4x + 8$

c $y = x^2 - 1$
 $3x + 2y = 0$



Example 17 Solving simultaneous equations with the quadratic formula

Solve the equations $y = x^2 + 5x - 5$ and $y = 2x$ simultaneously. Round your values to two decimal places.

SOLUTION

By substitution:

$$x^2 + 5x - 5 = 2x$$

$$x^2 + 3x - 5 = 0$$

Using the quadratic formula:

$$\begin{aligned}x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{9 + 20}}{2} \\ &= \frac{-3 \pm \sqrt{29}}{2} \\ &= 1.19258... \text{ or } -4.19258...\end{aligned}$$

In exact form, $y = 2x$

$$\begin{aligned}&= 2 \times \left(\frac{-3 \pm \sqrt{29}}{2} \right) \\ &= -3 \pm \sqrt{29}\end{aligned}$$

$\therefore$ The solutions are $x = 1.19, y = 2.39$
and $x = -4.19, y = -8.39$ (to 2 d.p.).

EXPLANATION

Rearrange into standard form.

$x^2 + 3x - 5$ does not factorise with whole numbers.

Quadratic formula: If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here, $a = 1$, $b = 3$ and $c = -5$.

Use a calculator to evaluate $\frac{-3 - \sqrt{29}}{2}$

and $\frac{-3 + \sqrt{29}}{2}$. Recall that if the number under the square root is negative, then there will be no real solutions.

Substitute exact x -values into $y = 2x$.

Round your values to two decimal places, as required.

Now you try

Solve the equations $y = x^2 + 2x - 1$ and $y = 3x$ simultaneously. Round your values to two decimal places.

**Example 18 Determining the number of solutions of simultaneous equations**

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 3x - 1$
 $y = x - 2$

b $y = 2x^2 - 3x + 8$
 $y = 5 - 2x$

SOLUTION

a By substitution:

$$x^2 + 3x - 1 = x - 2$$

$$x^2 + 2x + 1 = 0$$

Using the discriminant:

$$\begin{aligned} b^2 - 4ac &= (2)^2 - 4(1)(1) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

$\therefore$ There is one solution to the pair of equations.

b By substitution:

$$2x^2 - 3x + 8 = 5 - 2x$$

$$2x^2 - x + 3 = 0$$

Using the discriminant:

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(2)(3) \\ &= 1 - 24 \\ &= -23 < 0 \end{aligned}$$

$\therefore$ There is no solution to the pair of equations.

EXPLANATION

Start as if solving the equations simultaneously.

Once the equation is in the form $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ can be used to determine the *number* of solutions. Here, $a = 1$, $b = 2$ and $c = 1$.

Recall: $b^2 - 4ac > 0$ means two solutions.

$b^2 - 4ac = 0$ means one solution.

$b^2 - 4ac < 0$ means no solutions.

Substitute and rearrange into the form

$$ax^2 + bx + c = 0.$$

Calculate the discriminant. Here, $a = 2$, $b = -1$ and $c = 3$.

$b^2 - 4ac < 0$ means no solutions.

Now you try

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 7x - 4$
 $y = x - 1$

b $y = 3x^2 - x + 6$
 $y = 5 - 2x$

Exercise 7G

FLUENCY

$1-3(\frac{1}{2}), 4$

$1-3(\frac{1}{2}), 4, 5(\frac{1}{2})$

$1-3(\frac{1}{3}), 4, 5(\frac{1}{3})$

Example 15

- 1 Find the points of intersection of these parabolas and horizontal lines.

a $y = x^2 + x$
 $y = 6$

b $y = x^2 - 4x$
 $y = 12$

c $y = x^2 + 3x + 6$
 $y = 1$

d $y = 2x^2 + 7x + 1$
 $y = -2$

e $y = 4x^2 - 12x + 9$
 $y = 0$

f $y = 3x^2 + 2x + 9$
 $y = 5$

Example 16a,b

- 2 Solve these simultaneous equations using substitution.

a $y = x^2$
 $y = 3x$

b $y = x^2$
 $y = -2x$

c $y = x^2$
 $y = 3x + 18$

d $y = x^2 - 2x + 5$
 $y = x + 5$

e $y = -x^2 - 11x + 4$
 $y = -3x + 16$

f $y = x^2 + 3x - 1$
 $y = 4x + 5$

g $y = x^2 - 2x - 4$
 $y = -2x - 5$

h $y = -x^2 + 3x - 5$
 $y = 3x - 1$

i $y = 2x^2 + 4x + 10$
 $y = 1 - 7x$

j $y = 3x^2 - 2x - 20$
 $y = 2x - 5$

k $y = -x^2 - 4x + 3$
 $y = 2x + 12$

l $y = x^2 + x + 2$
 $y = 1 - x$

Example 16c

- 3 Solve these simultaneous equations by first substituting.

a $y = x^2$
 $2x + y = 8$

b $y = x^2$
 $x - y = -2$

c $y = x^2$
 $2x + 3y = 1$

d $y = x^2 + 3$
 $5x + 2y = 4$

e $y = x^2 + 2x$
 $2x - 3y = -4$

f $y = -x^2 + 9$
 $6x - y = 7$

Example 17



- 4 Solve the following simultaneous equations, making use of the quadratic formula.

- a** Give your answers to one decimal place where necessary.

i $y = 2x^2 + 3x + 6$
 $y = x + 4$

ii $y = x^2 + 1$
 $y = 2x + 3$

iii $y = -2x^2 + x + 3$
 $y = 3x + 2$

iv $y = 2x^2 + 4x + 5$
 $y = 3 - 2x$

- b** Give your answers in exact surd form.

i $y = x^2 + 2x - 5$
 $y = x$

ii $y = x^2 - x - 1$
 $y = 2x$

iii $y = -x^2 - 3x + 3$
 $y = -2x$

iv $y = x^2 + 3x - 3$
 $y = 2x + 1$

Example 18

- 5 Determine the number of solutions to the following simultaneous equations.

a $y = x^2 + 2x - 3$
 $y = x + 4$

b $y = 2x^2 + x$
 $y = 3x - 1$

c $y = 3x^2 - 7x + 3$
 $y = 1 - 2x$

d $y = x^2 + 5x + 1$
 $y = 2x - 3$

e $y = -x^2$
 $y = 2x + 1$

f $y = -x^2 + 2x$
 $y = 3x - 1$

6, 7($\frac{1}{2}$)

6, 7($\frac{1}{2}$)



-

- $$\mathbf{d} \quad y = \frac{8 - x^2}{2}$$
- $$y = 2(x - 1)$$

- 

- 9

9, 10

10, 11

- iii not at all.

- iii not at all.

-

—

12

- C** If the value of m is changed so that the line now intersects the parabola in two places, what is the set of possible values for m ?

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Designer jeans

- 1 A popular pair of designer jeans sells for \$300. A franchise sells 1200 pairs a month. The company carried out some research and discovered that for every \$10 decrease in price it can sell 100 more pairs a month.

The franchise is interested in maximising profit based on possible changes to the price of the jeans.

- a If the jeans are sold for \$290 one month:
- how many sales are expected?
 - how much is made in sales (revenue)? How does this compare to the revenue in a month when the jeans are sold for \$300?
- b Complete a table like the one shown below, to look at the revenue as the price of jeans is decreased.

Number of \$10 price decreases	Price of jeans (\$)	Number of sales	Revenue (\$)
0	300	1200	$300 \times 1200 = 360\,000$
1	290		
2			
3			

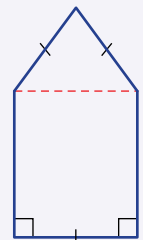
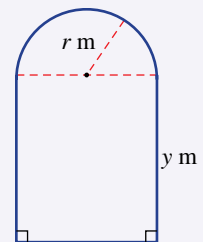
- c Use your table to help establish a rule for the revenue, R dollars, based on the number of \$10 price decreases, x .
- d Use your rule to determine the price to sell the jeans for to maximise the revenue and state this maximum revenue.

Stained-glass windows

- 2 Many objects, such as windows, are composite shapes. The stained-glass window shown, for example, is made up of a rectangle and semicircle.

A window company is interested in exploring the relationship between the perimeter of the window and its area. It also wants to look at maximising the area for a fixed perimeter.

- a If the perimeter of the window is fixed at 6 m, find:
- an expression for the height of the rectangular section, y m, in terms of r
 - a rule for the area, A m², of the window in the form $A = ar^2 + br$ where a and b are constants
 - the dimensions of the window that maximise its area and the maximum area.
- b Repeat part a for a perimeter of P m. Confirm your result by checking your answer to part a iii with $P = 6$.
- c Investigate a second stained-glass window, as shown, with an equilateral triangle top section. Compare its maximum area with part a, for the same perimeter of 6 m.



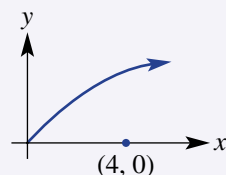
On the lake

- 3 The parabolic curve can be used to model many shapes in the natural world. A certain lake is represented on a Cartesian plane by the region bound by the curves $y = \frac{1}{2}x^2 - 4x$ and $y = 3x - \frac{1}{2}x^2$.

You will investigate the use of the parabola to model the boundary of the lake and a path of a boat. You will calculate distances on the lake and find minimum distances between objects on or beside the lake.

- a i Sketch the region represented by the lake including the points of intersection.
 ii At $x = 2$, determine the vertical distance across the lake.
 iii Find a rule in terms of x for the vertical distance (y direction) across the lake.
 iv Hence, find the maximum vertical distance across the lake.

- b A speed boat on the lake is following a path as shown on the right.
 A spectator stands on the sidelines at $(4, 0)$ with 1 unit representing 10 m.
 The rule for the speedboat's path is given by $y = \sqrt{x}$.



- i Find the direct distance, in metres, from the spectator to the speedboat when the speedboat is at $(1, 1)$.
 ii Find a rule in terms of x for the distance between $(4, 0)$ and a point (x, y) on the speed boat path. (*Hint: $y = \sqrt{x}$.*)
 iii As x increases, explain what happens to the value of $\sqrt{x}$.
 iv Complete the table below by using the minimums of the quadratics in rules i and iii to infer the minimum of their square root graph.

Rule	x -value of minimum	Minimum value
i $y = x^2 + 2$	0	2
ii $y = \sqrt{x^2 + 2}$		$\sqrt{2}$
iii $y = x^2 - 4x + 7$		
iv $y = \sqrt{x^2 - 4x + 7}$		

- v Hence, if the minimum value of a quadratic rule y is n at $x = m$, give the minimum value of $\sqrt{y}$ and for which x -value it occurs.
 vi Use the ideas above and your rule from part ii to determine the coordinates on the speedboat's path, $y = \sqrt{x}$, where it will be closest to the spectator. What is this minimum distance?



7H Rates of change

Learning intentions for this section:

- To be able to recognise and describe relationships between variables
- To understand the connection between gradient and the rate of change
- To be able to interpret distance–time graphs
- To know that speed is the rate of change of distance over time

Past, present and future learning:

- This section addresses the Stage 5 Path Topic *Variation and rates of change B*
- These concepts were not addressed in our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

We know from previous studies that many pairs of variables are related using linear, exponential or quadratic rules all of which produce different types of graphs. We note that the gradient of a linear graph is constant but for other relations the gradient will vary depending on the point on the curve. It is the gradient of the curve that tells us how quickly the value of y is changing. This is called the rate of change.



The rate of change of distance over time can also be considered as a gradient and is called speed.

Lesson starter: Movement graphs

This is a whole class activity. Two volunteers are needed: the ‘walker’ who completes a journey between the front and back of the classroom and the ‘grapher’ who graphs the journey on the whiteboard.

- 1 The ‘walker’ will complete a variety of journeys but these are not stated to the class. For example:
 - Walk slowly from the front of the room, stop halfway for a few seconds and then walk steadily to the back of the room. Stop and then return to the front at a fast steady pace.
 - Start quickly from the back of the room and gradually slow down until stopping at the front.
 - Start slowly from the front and gradually increase walking speed all the way to the back. Stop for a few seconds and then return at a steady speed to the front of the room.
- 2 The ‘grapher’ draws a graph of distance versus time on the whiteboard at the same time as the ‘walker’ moves. The distance is measured from the front of the room. No numbers are needed.
- 3 The class members also each draw their own distance–time graph as the ‘walker’ moves.
- 4 After each walk, discuss how well the ‘grapher’ has modelled the ‘walker’s’ movement.
- 5 This activity can also be done in reverse. The ‘grapher’ draws a distance–time graph on the board and the ‘walker’ moves to match the graph. The class checks that the ‘walker’ is following the graph correctly.

KEY IDEAS

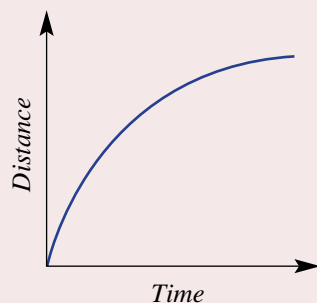
■ The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) varies.

- If y increases as x increases, then the rate of change (the gradient) is positive.
- If y decreases as x increases, then the rate of change (the gradient) is negative.
- If y does not change as x increases, then the rate of change (the gradient) is zero.
- Straight lines have a constant rate of change. There is a fixed change in y for each unit increase in x .
- Curves have a varying rate of change. The change in y varies for each unit increase in x .
- Analysing a graph and describing how both y and the rate of change of y varies allows us to check whether a given graph models a situation accurately.

For example, these distance–time graphs show various journeys from ‘home’ (distance = 0 at home).

Journey A

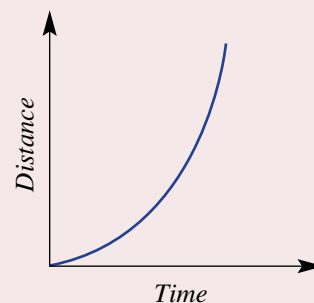
Decelerating away from home.



Distance from home is increasing at a decreasing rate.

Journey B

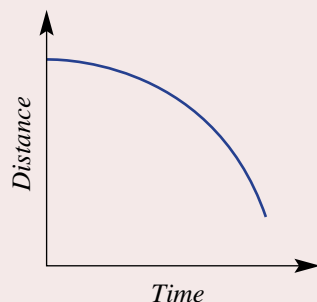
Accelerating away from home.



Distance from home is increasing at an increasing rate.

Journey C

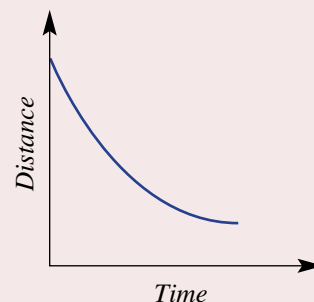
Accelerating towards home.



Distance from home is decreasing at an increasing rate.

Journey D

Decelerating towards home.

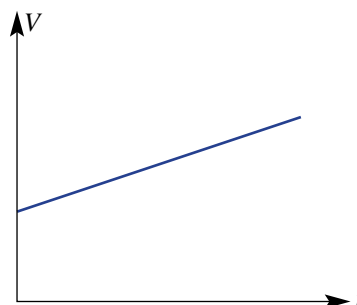


Distance from home is decreasing at a decreasing rate.

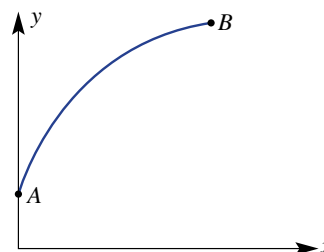
■ Speed is the rate of change of distance over time.

BUILDING UNDERSTANDING

- 1 This graph shows the relationship between two variables V and t .
- Would you say that the relationship between V and t is exponential, quadratic, linear or other?
 - Would you say that the gradient is constant as t varies? Give a reason.
 - How would you describe the rate of change of volume (V) over time in this example?



- 2 This graph shows the relationship between two variables x and y .
- At which point is the y -value the largest?
 - At which point is the gradient (rate of change) the greatest?
 - Are the y -values increasing or decreasing as x increases?
 - Is the gradient increasing or decreasing as x increases?

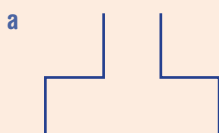


- 3 Choose a word to complete these sentences.
- If y increases as x increases, then the rate of change (gradient) is _____.
 - If y decreases as x increases, then the rate of change (gradient) is _____.
 - If y does not change as x increases, then the rate of change (gradient) is _____.

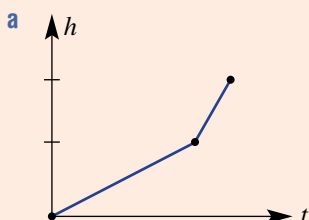


Example 19 Graphing height vs time

Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time, t .



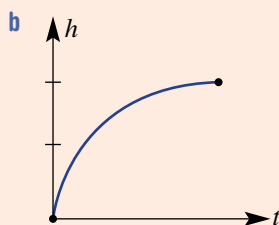
SOLUTION



EXPLANATION

For the wider part of the container, the rate of change of height (gradient) will be less than the rate of change for the top part of the container. It will take longer to fill the wider part of the container. For each section however the rate of change is constant.

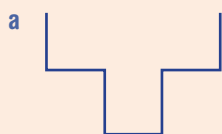
Continued on next page



As the water rises the rate of change (gradient) will decrease but still remain positive as the container becomes wider. The rate of change is greatest at the start and least at the end.

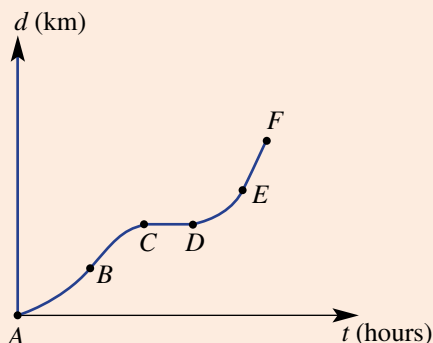
Now you try

Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h of water in the container at time, t .



Example 20 Working with distance–time graphs

This distance–time graph illustrates a journey by car over a number of hours.



- a** Between what points does the graph show a speed which is:
- i** zero?
 - ii** constant?
 - iii** increasing?
 - iv** decreasing?
- b** Would you say that the car is travelling slower at point B compared to point E ? Give a reason.
- c** Between the two given points, at what point is the car travelling the fastest?
- i** Points A and D
 - ii** Points C and E

SOLUTION

- a**
- i C to D
 - ii C to D and E to F
 - iii A to B and D to E
 - iv B to C
- b** Yes
- c**
- i B
 - ii E

EXPLANATION

Between points C and D the distance has not changed.

Between these pairs of points the gradient is constant.

Between these pairs of points the gradient is positive and increasing as the graph becomes steeper.

Between this pair of points the gradient is positive and decreasing as the graph becomes less steep.

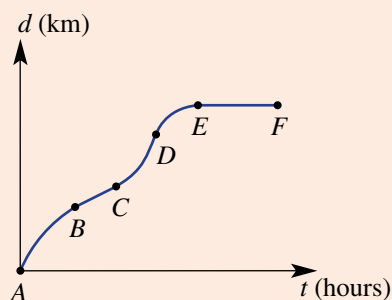
The graph looks steeper at point E compared to the graph at point B .

The graph is steepest at point B within this section.

The graph is steepest at point E within this section.

Now you try

This distance–time graph illustrates a journey by bicycle over a number of hours.



- a** Between what points does the graph show a speed which is:
- i zero?
 - ii constant?
 - iii increasing?
 - iv decreasing?
- b** Would you say that the bicycle is travelling slower at point B compared to point D ? Give a reason.
- c** Between the two given points, at what point is the bicycle travelling the fastest?
- i Points A and C
 - ii Points C and F

Exercise 7H

FLUENCY

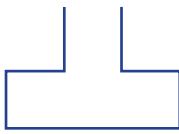
1–5

1–6

2, 3, 5, 6

- Example 19a** 1 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time, t .

a



b



c

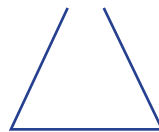


- Example 19b** 2 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time, t .

a



b

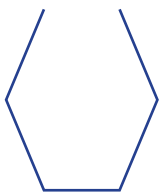


c

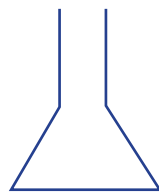


- 3 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time, t .

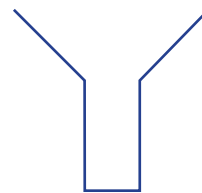
a



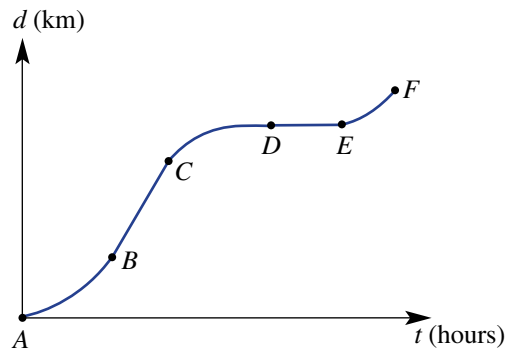
b



c

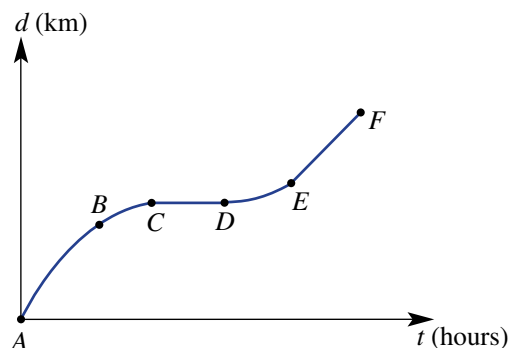


- Example 20** 4 This distance–time graph illustrates a journey by car over a number of hours.

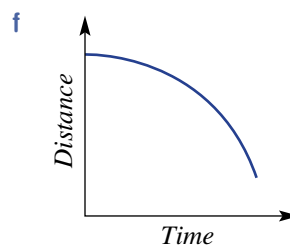
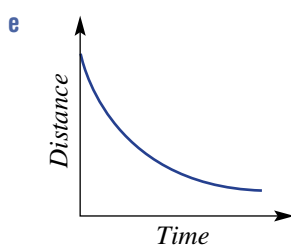
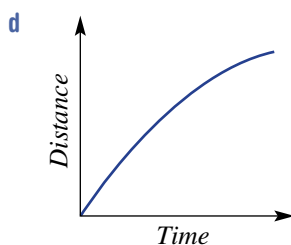
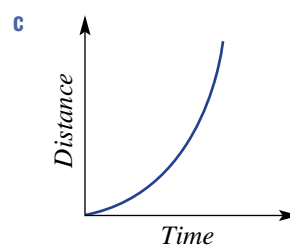
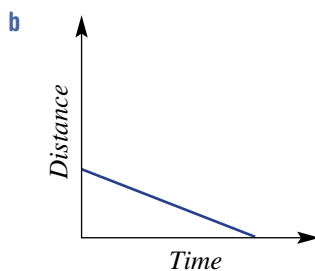
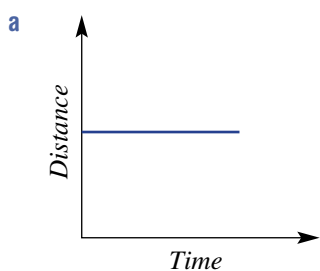


- a Between what points does the graph show a speed which is:
- zero?
 - constant?
 - increasing?
 - decreasing?
- b Would you say that the car is travelling slower at point B compared to point F ? Give a reason.
- c Between the two given points, at what point is the car travelling the fastest?
- Points A and B
 - Points C and E

- 5 This distance–time graph illustrates a journey by bicycle over a number of hours.



- a Between what points does the graph show a speed which is:
- zero?
 - constant?
 - increasing?
 - decreasing?
- b Would you say that the bicycle is travelling slower at point *B* compared to point *E*? Give a reason.
- c Between the two given points, at what point is the bicycle travelling the fastest?
- Points *B* and *D*
 - Points *C* and *E*
- 6 The distance–time graphs below show various journeys, each with distance measured from ‘home’ (i.e. distance = 0 at home). For each graph, select and copy one correct description from each category below of how the distance from home, the gradient and the speed are varying.



Distance from home

Increasing distance from home

Decreasing distance from home

Fixed distance from home

Gradient of graph

Positive constant gradient

Positive varying gradient

Negative constant gradient

Negative varying gradient

Zero gradient

Speed

Stationary

Lower constant speed

Higher constant speed

Decreasing speed, decelerating

Increasing speed, accelerating

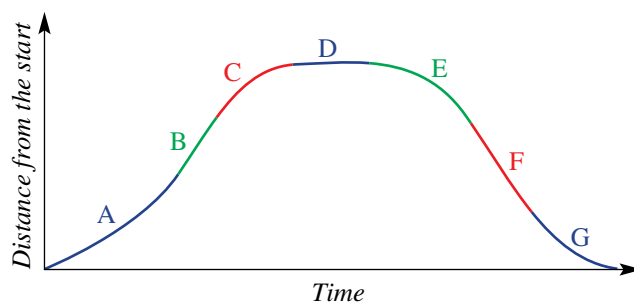
PROBLEM-SOLVING

7, 8

7–9

8–10

- 7 From the lists below, select and copy the correct description for the rate and speed of each segment of this distance–time graph. The rate of change of distance with respect to time is the gradient.

**Rate of change of distance with respect to time****Speed**

Positive constant rate of change

Stationary

Positive varying rate of change

Constant speed

Negative constant rate of change

Decreasing speed, decelerating

Negative varying rate of change

Increasing speed, accelerating

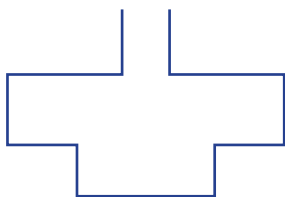
Zero rate of change

- 8 Sketch a population–time graph from each of these descriptions
- A population of bilbies is decreasing at a decreasing rate.
 - A population of Tasmanian devils is decreasing at an increasing rate.
 - A population of camels is increasing at a decreasing rate.
 - A population of rabbits is increasing at an increasing rate.



- 9 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time, t .

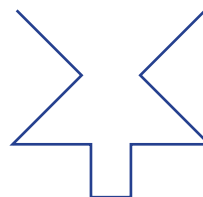
a



b



c



- 10 Draw a possible graph of speed, s , vs time, t , for an athlete running the following races.
- 100 metres
 - 5000 metres

REASONING

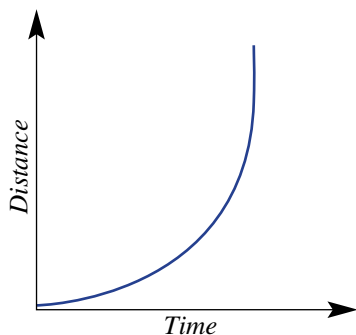
11

11, 12

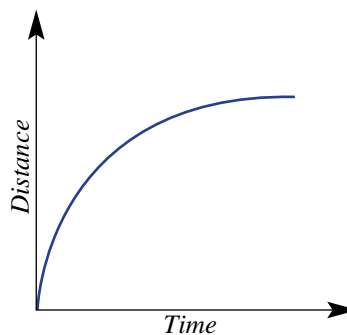
12, 13

- 11 Which of the following graphs don't match the journey description correctly or are not physically possible? For each graph, explain the feature that is incorrect and redraw it correctly. Distances are to be interpreted as distance from the original starting point (displacement).

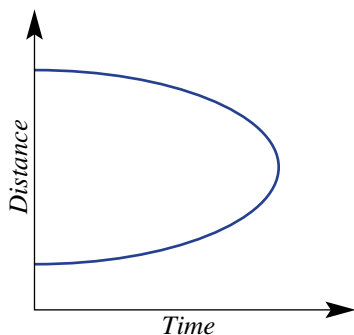
- a Stopped, then accelerating to a very high speed.



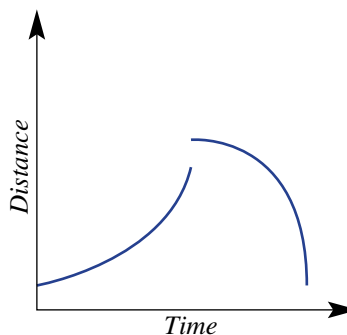
- b Moving at very high speed and decelerating to a stop.



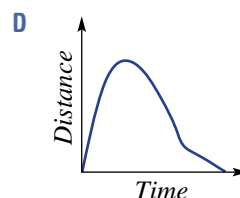
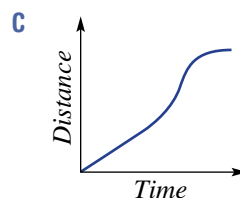
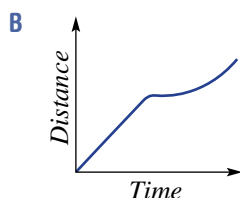
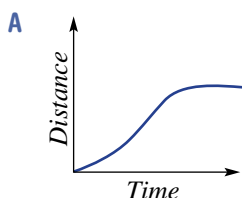
- c Accelerating, changing direction, then decelerating.



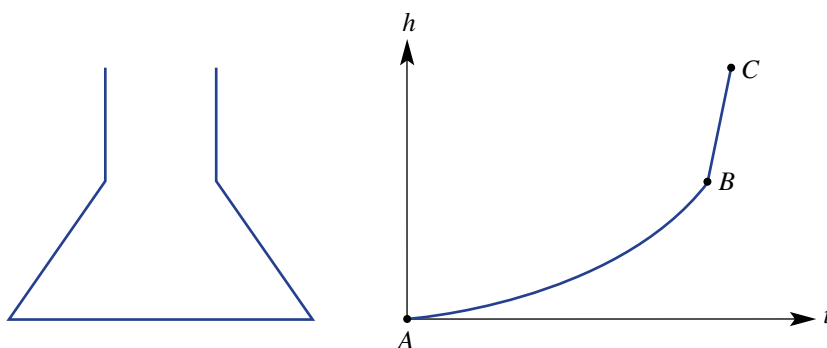
- d Accelerating, changing direction, then decelerating.



- 12 Match each distance–time graph (A–D) with the correct journey described (a–d). Give reasons for each choice, describing how the changing distance and varying rate relates to the movement of the object.



- a** A soccer player runs at a steady speed across the field, stops briefly to avoid a tackle and then accelerates farther away.
 - b** A rocket stage 1 booster accelerates to huge speed, then detaches and quickly decelerates, then accelerates as it falls towards Earth, finally a parachute opens and it slows, falling to Earth at a steady speed.
 - c** A motorbike moves at a steady speed, then accelerates to pass a car, then brakes and decelerates, coming to a stop at traffic lights.
 - d** A school bus accelerates away from the bus stop, then moves at a steady speed and then decelerates and stops as it arrives at the next bus stop.
- 13 When drawing a graph of height (h) vs time (t) for the level of water in this container as it is being filled, a student draws the following graph.



- a** Describe the error in the graph at the point B.
- b** Draw a corrected graph.

Enrichment: Creating distance–time graphs

–

–

14

- 14 Work in small groups to develop distance–time graphs from recorded data. Equipment: 100 m tape measure, stopwatch, recording materials, video camera.
- a** Determine a suitable method for recording the distance a student has moved after every 5 seconds over a 30 second period.
 - b** Select a variety of activities for the moving student to do in each 30 second period. For example, walking slowly, running fast, starting slowly then speeding up, etc.
 - c** Record and graph distance versus time for each 30 second period.
 - d** For each graph, using sentences with appropriate vocabulary, describe how the distance and rate of change of distance is varying.
 - e** Analyse how accurately each graph has modelled that student's movement.

71 Average and instantaneous rates of change of change EXTENDING

Learning intentions for this section:

- To understand the difference between average and instantaneous rates of change
- To be able to calculate the average rate of change including average speed
- To be able to approximate an instantaneous rate of change using an average rate of change

Past, present and future learning:

- This section goes beyond the Stage 5 Path Topic *Variation and rates of change B*
- These concepts were not addressed in our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended

When dealing with a linear relation, we know the exact rate of change at all points because the gradient is constant. For other relations however the rate of change will vary across the range of points. The gradient at a particular point is called the instantaneous rate of change and can be illustrated using a tangent passing through that point. The gradient of this tangent cannot be calculated easily using $m = \frac{\text{rise}}{\text{run}}$ because only one point is known on the tangent. We therefore use the average rate of change between two selected points to approximate the instantaneous rate of change.

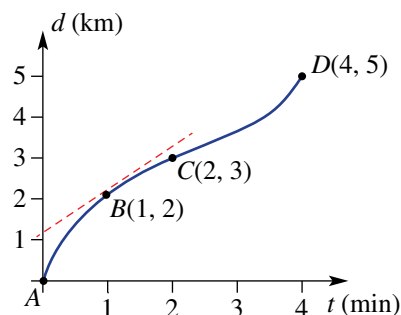


The average rate of change can be represented as a straight-line graph joining two points.

Lesson starter: Comparing average and instantaneous rate of change

Consider this distance–time graph.

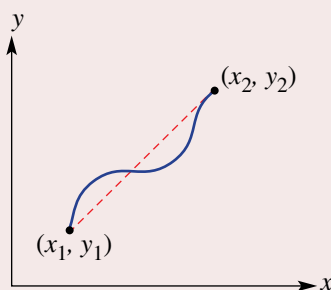
- a** Find the gradient between the following pairs of points.
- | | | |
|--------------|--------------|---------------|
| i AB | ii AC | iii AD |
| iv BC | v BD | vi CD |
- b** Knowing that the average rate of change between two points can be found by finding the gradient, find the average rate of change between these pairs of points. Units will be km/min.
- | | |
|-------------|--------------|
| i AC | ii AD |
|-------------|--------------|
- c** The instantaneous rate of change at *B*, shown using a dashed tangent line, can be approximated using the gradients of *AB*, *BC* or *AC*. Which of these gradients would give the best approximation? Give a reason.



KEY IDEAS

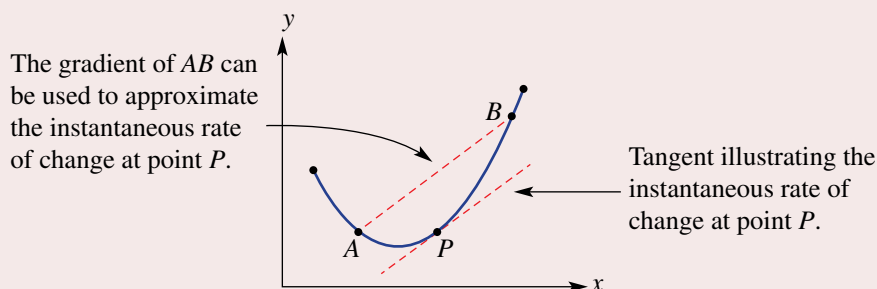
- The **average rate of change** of a variable y as x varies between two points is equal to the gradient of the line joining the same two points.

• Average rate of change $= \frac{y_2 - y_1}{x_2 - x_1}$



- The **instantaneous rate of change** is equal to the gradient at a given point.

- The instantaneous rate of change can be approximated by the average rate of change between two selected points.

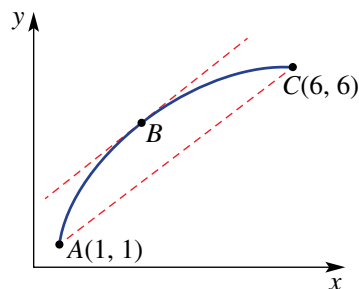


- Average speed is equal to the average rate of change of distance over time.

BUILDING UNDERSTANDING

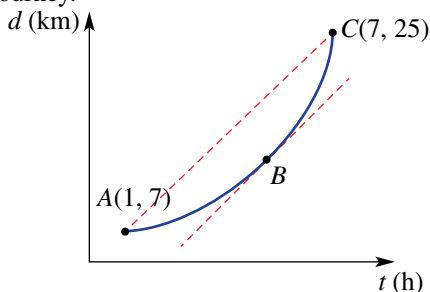
- 1 Consider this curve joining the points $A(1, 1)$ and $C(6, 6)$.

- Find the gradient of the dashed line segment between the points A and C .
- State the average rate of change between the points A and C .
- Which line segment AB , BC or AC gives the best approximation of the instantaneous rate of change at point B ? Give a reason.



- 2 This graph shows a distance–time graph for part of a journey. Consider this curve joining the points $A(1, 7)$ and $C(7, 25)$.

- Find the gradient of the dashed line segment between the points A and C .
- State the average speed between the points A and C .
- Which line segment AB , BC or AC gives the best approximation of the instantaneous rate of change (speed) at point B ? Give a reason.





Example 21 Finding an average rate of change

Consider the relation with rule $y = x^2 - 3$. Find the average rate of change of y as x changes from:

a 1 to 4

b 1 to 2

SOLUTION

a At $x = 1$, $y = (1)^2 - 3 = -2$

At $x = 4$, $y = (4)^2 - 3 = 13$

$$\begin{aligned}\text{Average rate of change} &= \frac{13 - (-2)}{4 - 1} \\ &= \frac{15}{3} \\ &= 5\end{aligned}$$

b At $x = 1$, $y = (1)^2 - 3 = -2$

At $x = 2$, $y = (2)^2 - 3 = 1$

$$\begin{aligned}\text{Average rate of change} &= \frac{1 - (-2)}{2 - 1} \\ &= \frac{3}{1} \\ &= 3\end{aligned}$$

EXPLANATION

First find the coordinates of the two points using substitution.

Use $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient which gives the average rate of change.

The y -values at $x = 1$ and $x = 2$ are -2 and 1 respectively.

The gradient gives the average rate of change.

Now you try

Consider the relation with rule $y = -x^2 + 2x$. Find the average rate of change of y as x changes from:

a 0 to 5

b 0 to 1



Example 22 Approximating an instantaneous rate of change

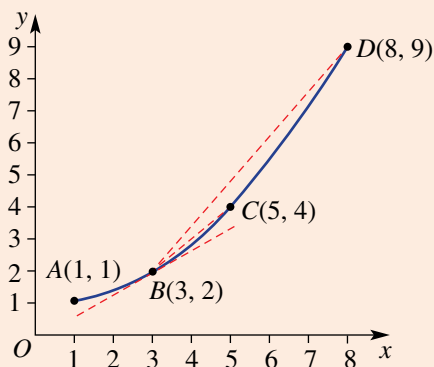
This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

i BC

ii BD

b Which of the line segments BC or BD gives a better approximation of the instantaneous rate of change at point B ? Give a reason.



Continued on next page

a Gradient $BC = \frac{4-2}{5-3} = 1$

$$\text{Gradient } BD = \frac{9-2}{8-3} = \frac{7}{5}$$

- b** BC gives a better approximation as the gradient of BC is closer to the instantaneous rate of change at B .

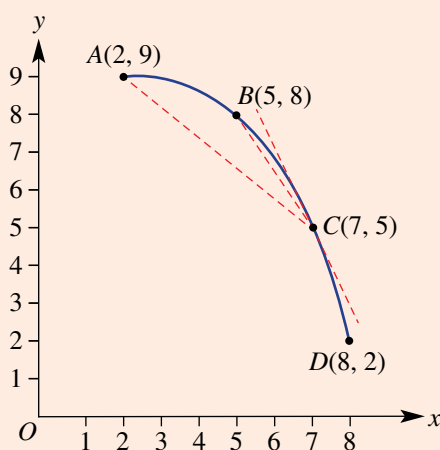
Use $m = \frac{\text{rise}}{\text{run}}$ to find the gradient of a line segment.

Use $m = \frac{\text{rise}}{\text{run}}$ to find the gradient of a line segment.

Looking at the given graph, the gradient of BD is greater than the gradient of BC , which is higher but closer to the gradient of the tangent at B .

This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at that point.

- a** Find the gradient of the following line segments.
- i** AC **ii** BC
- b** Which of the line segments AC or BC gives a better approximation of the instantaneous rate of change at point C ? Give a reason.



Exercise 71

FLUENCY

1-3, 5, 6

1, 3–6

2, 3, 5, 7

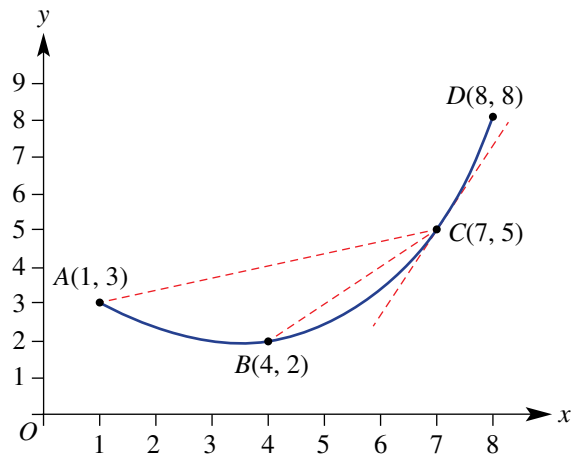
Example 21

- 1 Consider the relation with rule $y = x^2 + 2$. Find the average rate of change of y as x changes from:
a 1 to 3 **b** 1 to 2
- 2 Consider the relation with rule $y = -x^2 + 4x$. Find the average rate of change of y as x changes from:
a 0 to 4 **b** 0 to 1
- 3 Consider the relation with rule $y = x^2 - x + 3$. Find the average rate of change of y as x changes from:
a -1 to 3 **b** -1 to 0
- 4 Consider the relation with rule $y = -2x^2 + 3x - 1$. Find the average rate of change of y as x changes from:
a -2 to 4 **b** -2 to -1

Example 22

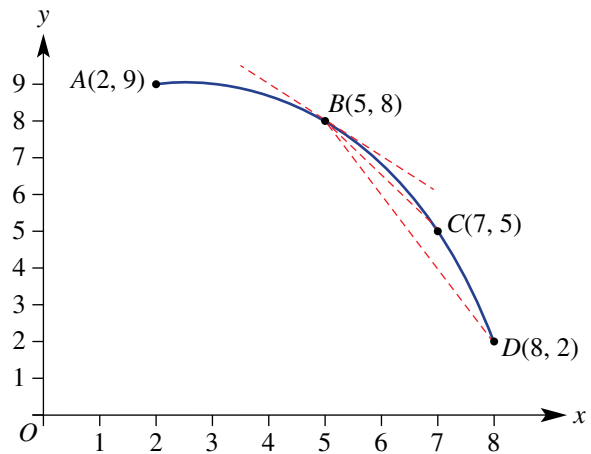
- 5 This graph shows part of a curve with a tangent drawn at point C to illustrate the instantaneous rate of change at that point.

- a Find the gradient of the following line segments.
- AC
 - BC
- b Which of the line segments AC or BC gives a better approximation of the instantaneous rate of change at point C ? Give a reason.



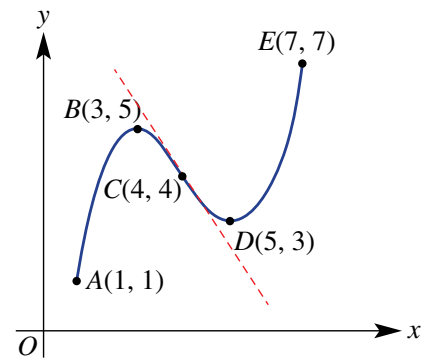
- 6 This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at that point.

- a Find the gradient of the following line segments.
- BC
 - BD
- b Which of the line segments BC or BD gives a better approximation of the instantaneous rate of change at point B ? Give a reason.



- 7 This graph shows part of a curve with a tangent drawn at point C to illustrate the instantaneous rate of change at that point.

- a Find the gradient of the following line segments.
- BD
 - BE
- b Which of the line segments BD or BE gives a better approximation of the instantaneous rate of change at point C ? Give a reason.



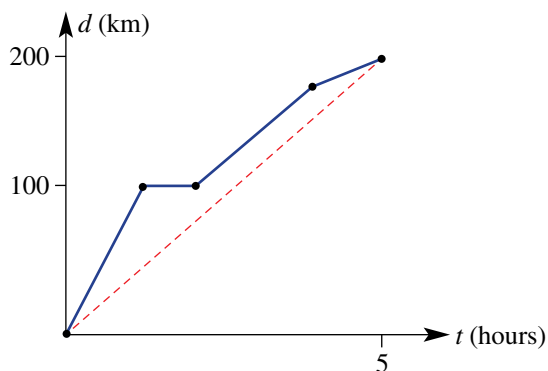
PROBLEM-SOLVING

8, 9

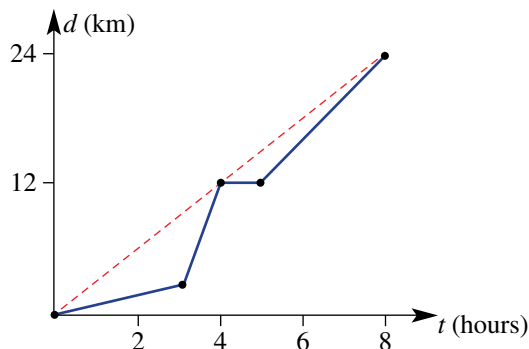
8–10

9–11

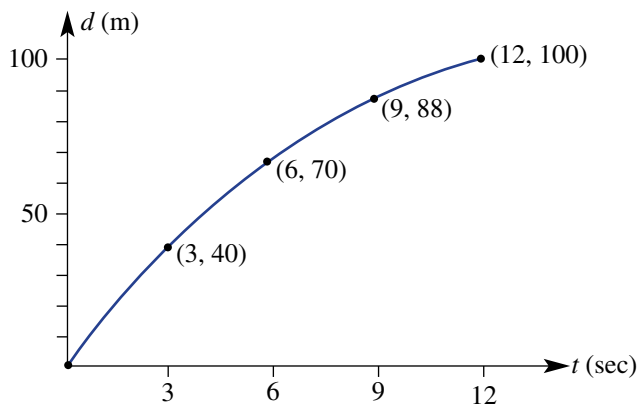
- 8 A journey by a vehicle covers 200 km in 5 hours as shown. Find the average rate of change of distance (average speed) over the 5 hours.



- 9 A journey by a hiker covers 24 km in 8 hours as shown. Find the average rate of change of distance (average speed) over the 8 hours.



- 10 A sprinter's distance-time graph for a 100 m race is shown here. Use the points (3, 40) and (9, 88) to help estimate the sprinter's speed at the 6 second mark.



- 11 The volume of liquid in a leaking bucket is modelled by the rule $V = \frac{2}{5}(t - 5)^2$ for $0 \leq t \leq 5$ where V is in litres and t is in seconds.
- Find the volume of liquid in the bucket at:
 - $t = 0$
 - $t = 5$.
 - Find the average rate of change of volume over the 5 seconds.
 - Estimate the instantaneous rate of change of volume after:
 - 3 seconds
 - 1 second.

REASONING

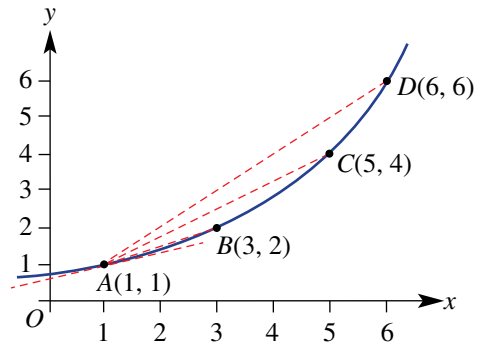
12

12, 13

13, 14

- 12 This graph shows a curve passing through four points and a tangent at point A.

- a Which segment AB, AC or AD would provide the best approximation to the gradient of the tangent at A? Give a reason.
- b Use your answer from part a to estimate the instantaneous rate of change at point A.



- 13 A curve has rule $y = x^2 - 1$.

- a Find the average rate of change between the points with the following x -values.
- i 1 and 2 ii 1 and 1.5 iii 1 and 1.1
- b Which pair of x -values from part a would give the best approximation of the instantaneous rate of change at $x = 1$? Give a reason.
- c Explain how you might find an even better approximation of the instantaneous rate of change at $x = 1$.

- 14 One way to approximate the instantaneous rate of change at a point $x = a$ is to choose two points P and Q , either side of that point. P and Q are chosen so that the average of their x -values is equal to a . We then find the average rate of change between points P and Q .

Use this idea to approximate the instantaneous rate of change of the following relations at the given point.

- a $y = 2^x$, ($x = 1$) b $y = \frac{1}{x}$, ($x = -2$)
- c $y = x^3$, ($x = -2$) d $y = \sqrt{x}$, ($x = 2$)

ENRICHMENT: When an approximate rate of change gives the exact instantaneous rate of change

—

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15($\frac{1}{2}$)

- 15 One interesting fact about quadratic relations is that the instantaneous rate of change at $x = a$ is equal to the average rate of change between the points where $x = a - h$ and $x = a + h$ where h is any chosen number.

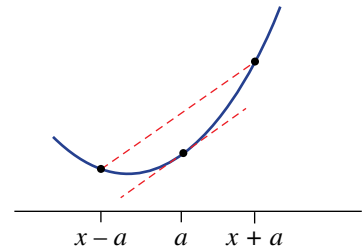
For example: For $y = x^2$ the instantaneous rate of change at $x = 2$ can be found using $h = 1$ by finding the gradient between the point at $x = 2 - 1 = 1$ and $x = 2 + 1 = 3$.

$$m = \frac{9 - 1}{3 - 1} = 4$$

So the instantaneous rate of change at $x = 2$ is 4.

Use this result to find the instantaneous rate of change of the following quadratics at the given points.

- a $y = x^2$, ($x = 3$) b $y = 2x^2 - 1$, ($x = 1$)
- c $y = -x^2 + 4$, ($x = 2$) d $y = -3x^2 + 2x - 1$, ($x = -1$)
- e $y = -2x^2 + 5x - 3$, ($x = -1$) f $y = 4(x - 2)^2 - 5$, ($x = -2$)



7J Direct variation and inverse variation

Learning intentions for this section:

- To understand the relationship between two variables that are directly proportional or inversely proportional
- To understand that the shape of a graph shows how a variable and its rate of change varies
- To be able to find and use rules involving direct and inverse proportion
- To know how to model and interpret a situation using a distance–time graph

Past, present and future learning:

- This section addresses the Stage 5 Path Topic called *Variation and rates of change A*
- Some of these concepts were not addressed in our Year 9 book
- In Stage 6 Advanced these concepts will be revised, briefly, then extended
- You may be expected to recall the formulas from memory in the HSC Examination

Two variables are said to be directly related if they are in a constant (i.e. unchanged) ratio. If two variables are in direct proportion, as one variable increases so does the other. For example, consider the relationship between speed and distance travelled in a given time. In 1 hour, a car can travel 50 km at 50 km/h, 100 km at 100 km/h, etc.

For two variables in inverse or indirect variation, as one variable increases the other decreases. For example, consider a beach house that costs \$2000 per week to rent. As the number of people renting the house increases, then the cost per person decreases.

Lesson starter: Birthday party relationships

For a birthday party, tickets are purchased at an activity centre for \$22 each.

- a** Describe the relationship between the total cost of the tickets \$ C and the number of tickets purchased, n . Express this as a rule.

- b** Use your rule to find the total cost of purchasing 8 tickets.

At the end of the party, a 2 kg bag of chocolate is distributed evenly among all the friends.

- c** Describe the relationship between the amount of chocolate

received by each person and the number of friends at the party. Express this as a rule.

- d** Use your rule to find the amount of chocolate received by each person if there are 8 friends.



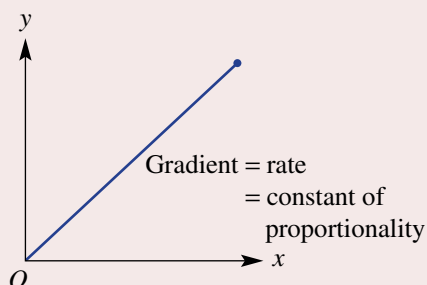
Renting a holiday house can be expensive. But, when sharing, the rent per person is inversely proportional to the number of people. However, the total rent increases in direct proportion to the length of stay.

KEY IDEAS

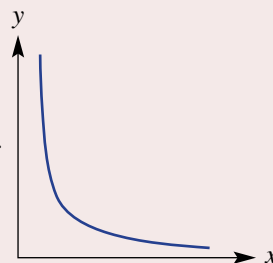
- y varies directly with x if their relationship is

$$y = kx \text{ or } \frac{y}{x} = k.$$

- k is a constant, and is called the constant of proportionality.
- The graph of y versus x gives a straight line that passes through the origin, O or $(0, 0)$, where k is the gradient.

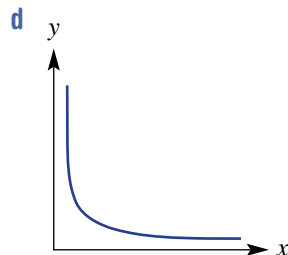
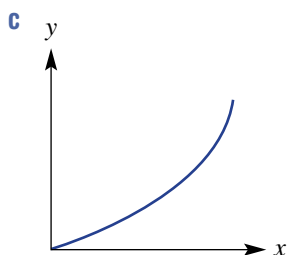
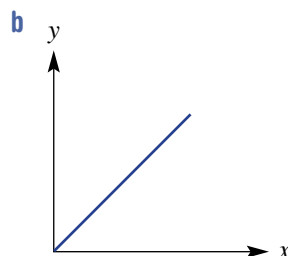
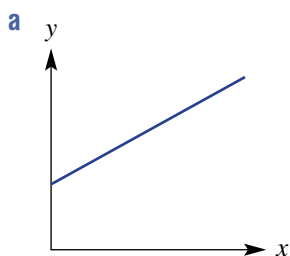


- We write: $y \propto x$ which means that $y = kx$.
 - We say: y varies directly as x , or y is **directly proportional** to x .
- y varies inversely with x if their relationship is $y = \frac{k}{x}$ or $xy = k$.
- The graph of y versus x gives a shape called a hyperbola.
 - We write: $y \propto \frac{1}{x}$, which means that $y = \frac{k}{x}$ or $xy = k$.
 - We say: y varies inversely as x , or y is **inversely proportional** to x .



BUILDING UNDERSTANDING

- 1 For each pair of variables, state whether they are in direct or inverse proportion or neither.
- The number of *hours* worked and *wages* earned at a fixed rate per hour.
 - The *volume* of remaining fuel in a car and the *cost* of filling the fuel tank.
 - The *speed* and *time* taken to drive a certain distance.
 - The *size* of a movie file and the *time* for downloading it to a computer at a constant rate of kB/s.
 - The *cost* of a taxi ride and the *distance* travelled. The cost includes flag fall (i.e. a starting charge) and a fixed \$/km.
 - The *rate* of typing in words per minute and the *time* needed to type a particular assignment.
- 2 State the main features of each graph and whether it shows direct proportion or inverse (i.e. indirect) proportion or neither.





Example 23 Finding and using a direct variation rule

If m is directly proportional to h and $m = 90$ when $h = 20$, determine:

- a** the relationship between m and h
b m when $h = 16$
c h when $m = 10$.

SOLUTION

$$\begin{aligned}\mathbf{a} \quad m &\propto h \\ m &= kh \\ 90 &= k \times 20 \\ k &= \frac{90}{20} \\ k &= \frac{9}{2} \\ \therefore m &= \frac{9}{2}h\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad m &= \frac{9}{2} \times 16 \\ \therefore m &= 72\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 10 &= \frac{9}{2} \times h \\ 10 \times \frac{2}{9} &= h \\ \therefore h &= \frac{20}{9}\end{aligned}$$

EXPLANATION

First, write the variation statement.

Write the equation, including k .

Substitute $m = 90$ and $h = 20$.

Divide both sides by 20 to find k .

Simplify.

Write the rule using the value of k found.

Substitute $h = 16$.

Simplify to find m .

Substitute $m = 10$.

Solve for h .

Now you try

If a is directly proportional to b and $a = 120$ when $b = 80$, determine:

- a** the relationship between a and b
b a when $b = 40$
c b when $a = 12$.



Example 24 Finding and using an inverse proportion rule

If x and y are inversely proportional and $y = 6$ when $x = 10$, determine:

- a** the constant of proportionality, k , and write the rule
b y when $x = 15$
c x when $y = 12$.

SOLUTION

$$\begin{aligned}\mathbf{a} \quad y &\propto \frac{1}{x} \\ y &= \frac{k}{x} \\ k &= xy \\ k &= 10 \times 6 \\ k &= 60 \\ y &= \frac{60}{x}\end{aligned}$$

EXPLANATION

Write the variation statement.

Write the equation, including k .

The constant of proportionality, $k = xy$.

Use $x = 10$, $y = 6$.

Substitute $k = 60$ into the rule $y = \frac{k}{x}$.

- 3 A vehicle drives at 80 km/h.
- Find a rule linking the distance that the vehicle travels, d km, after t hours.
 - Would you say that the distance d is directly proportional or inversely proportional to the time t ?
 - Find:
 - how far the vehicle travels after 4 hours
 - how long it takes for the vehicle to travel 200 km.
- 4 A cyclist completes a 20 km journey at an average speed of s km/h and takes t hours.
- Find a rule linking the average speed that the bike travels, s km/h, after t hours.
 - Would you say that the speed s is directly proportional or inversely proportional to the time t ?
 - Find:
 - the average speed required if the journey is to take 2.5 hours
 - the time taken if the average speed is 10 km/h.
- 5 The amount that a farmer earns from selling wheat is in direct proportion to the number of tonnes harvested.
- Find the constant of proportionality, k , given that a farmer receives \$8296 for 34 tonnes of wheat.
 - Write the direct proportion equation relating selling price, P , and number of tonnes, n .
 - Calculate the selling price of 136 tonnes of wheat.
 - Calculate the number of tonnes of harvested wheat that is sold for \$286 700.

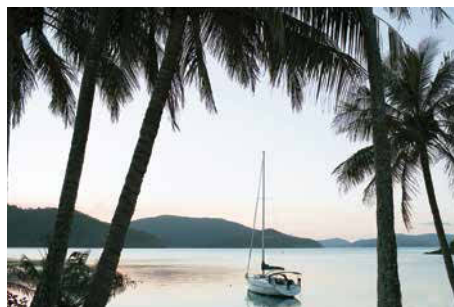
PROBLEM-SOLVING

6

6, 7

6, 7

- 6 A 30-seater school bus costs 20 students \$3.70 each to hire for a day. The overall cost of the bus remains the same regardless of the number of students.
- Write a relationship between the cost per student (c) and the number of students (s).
 - If only 15 students use the bus, what would be their individual cost, to the nearest cent?
 - What is the minimum a student would be charged, to the nearest cent?
- 7 For each relationship described below:
- write a suitable equation
 - sketch the graph, choosing appropriate values for the initial and final points on the graph.
- The distance that a car travels in 1 hour is directly proportional to the speed of the car. The roads have a 100 km/h speed limit.
 - The cost per person of hiring a yacht is inversely proportional to the number of people sharing the total cost. A yacht in the Whitsunday Islands can be hired for \$320 per day for a maximum of eight people on board.
 - There is a direct proportional relationship between a measurement given in metric units and in imperial units. A weight measured in pounds is 2.2 times the value of the weight in kilograms.
 - The time taken to type 800 words is inversely proportional to the typing speed in words per minute.



REASONING

8

8, 9

9, 10

- 8 Decide if these pairs of variables are in direct proportion (*D*) or inverse proportion (*I*).

- a Amount earned and Number of hours worked
- b Amount of work completed and Test grades
- c Time to complete a job and Number of helpers
- d Speed and Time
- e Distance and Time
- f Distance and Speed
- g Profit and Number of sales
- h Time to paint a house and Number of painters

- 9 Partial variation occurs when one variable is partly constant and partly varies with another variable. For example: The cost of producing n cakes in a day's work at a bakery involves a fixed cost of \$200 plus a 50 cent cost per cake.

- a Find a rule connecting the total cost \$ C of producing n cakes.
- b Find the total cost of producing 60 cakes.
- c Find how many cakes can be produced if the total cost is \$242.50.



- 10 Joint variation occurs when one variable varies directly with two or more variables. The area of a triangle, for example, varies directly with the base length b and the height h . The rule connecting the variables is $A = \frac{1}{2}bh$.

- a What is the constant of proportionality in the above given example?
- b Does the value of A increase or decrease as b increases?
- c Does the value of A increase or decrease as h decreases?

ENRICHMENT: Combining variables with jigsaw puzzles

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11

- 11 Combined variation involves a variable y , for example, varying directly with variable x and inversely with variable z . A rule of such variation is of the form $y = \frac{kx}{z}$.

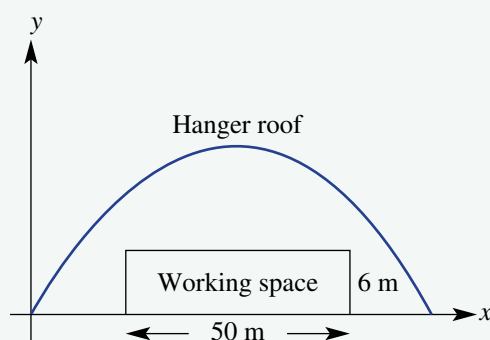
The time it takes to complete a jigsaw puzzle, t hours, for example, depends on the number of pieces, p , and the number of people working on the puzzle, n .

- a Write a rule for t in terms of p and n and the constant of variation k .
- b Find the value of k if it takes 5 hours for 2 people to complete a 2000-piece puzzle.
- c Using your above results find:
 - i the time taken to complete a 5000-piece puzzle with 5 people.
 - ii the number of people required to complete a 10 000-piece puzzle in 5 hours.

Designing an aircraft hanger

As an engineer you are designing a curved parabolic roof to house a working space for the building of aircraft. The cross-section of the working space is to be 6 m in height and at least 50 m in breadth. The total breadth of the hanger cannot be more than 100 m.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.



Routine problems

- Sketch a graph of $y = 0.01x(80 - x)$ for $y > 0$, where x and y are in metres.
- Decide if the working space for the planes fits inside the model for the roof defined by the rule in part **a**. Give reasons.
- Repeat parts **a** and **b** above for $y = 0.002x(100 - x)$.

Non-routine problems

- The problem is to find a model for a hanger roof of minimum height to house the working space. Write down all the relevant information that will help solve this problem with the aid of a diagram.
- Choose at least three realistic values of a using the model $y = ax(80 - x)$ which will house the aircraft working space, 50 m wide and 6 m high. Justify your choices using graphs.
- Determine a value of a for the model $y = ax(80 - x)$ which minimises the height of the hanger.
- Choose your own values of a and b for the model $y = ax(b - x)$ which will:
 - house the aircraft working space, 50 m wide and 6 m high
 - ensure that the total breadth of the hanger is not more than 100 m.
- Determine the values of a and b for the model $y = ax(b - x)$ which minimises the total height of the hanger.
- Compare the height of the hanger for the different models chosen above.
- Explain why your choices for the values of a and b minimise the total height of the hanger.
- Summarise your results and describe any key findings.

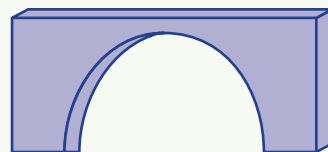
Extension problem

- Choosing the model $y = 0.01x(80 - x)$, determine the maximum possible breadth of the working space for a given working space height of 6 m.



Painting bridges

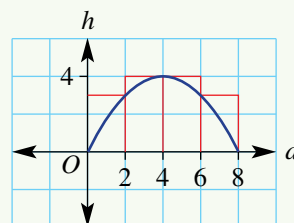
A bridge is 6 m high and has an overall breadth of 12 m and an archway underneath, as shown.



We need to determine the actual surface area of the face of the arch so that we can order paint to refurbish it. The calculation for this area would be:

$$\begin{aligned}\text{Bridge area} &= 12 \times 6 - \text{area under arch} \\ &= 72 - \text{area under parabola}\end{aligned}$$

Consider an archway modelled by the formula $h = -\frac{1}{4}(d - 4)^2 + 4$, where h metres is the height of the arch and d metres is the distance from the left. To estimate the area under the arch, divide the area into rectangular regions. If we draw rectangles above the arch and calculate their areas, we will have an estimate of the area under the arch even though it is slightly too large. Use the rule for h to obtain the height of each rectangle.



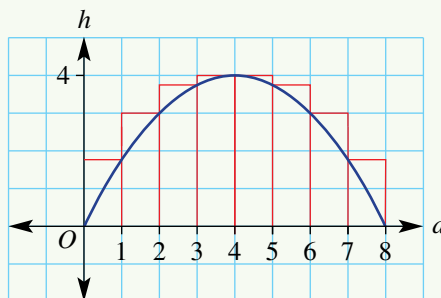
$$\begin{aligned}\text{Area} &= (2 \times 3) + (2 \times 4) + (2 \times 4) + (2 \times 3) \\ &= 6 + 8 + 8 + 6 \\ &= 28 \text{ m}^2\end{aligned}$$

$\therefore$ Area is approximately 28 m^2 .

We could obtain a more accurate answer by increasing the number of rectangles; i.e. by reducing the breadth of each rectangle (called the strip breadth).

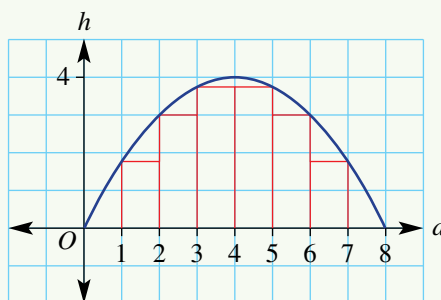
Overestimating the area under the arch

- Construct an accurate graph of the parabola and calculate the area under it using a strip breadth of 1.
- Repeat your calculations using a strip breadth of 0.5.
- Calculate the surface area of the face of the arch using your answer from part **b**.



Underestimating the area under the arch

- Estimate the area under the arch by drawing rectangles under the graph with a strip breadth of 1.
- Repeat the process for a strip breadth of 0.5.
- Calculate the surface area of the face of the arch using your answer from part **b**.



Improving accuracy

- Suggest how the results from the above parts could be combined to achieve a more accurate result.
- Explore how a graphics or CAS calculator can give accurate results for finding areas under curves.

- 1 Solve these inequalities for x .

a $6x^2 + x - 2 \leq 0$
 b $12x^2 + 5x - 3 > 0$
 c $x^2 - 7x + 2 < 0$

- 2 Prove the following.

- a The graphs of $y = x - 4$ and $y = x^2 - x - 2$ do not intersect.
 b The graphs of $y = -3x + 2$ and $y = 4x^2 - 7x + 3$ touch at one point.
 c The graphs of $y = 3x + 3$ and $y = x^2 - 2x + 4$ intersect at two points.

- 3 For what values of k does the graph of $y = kx^2 - 2x + 3$ have:

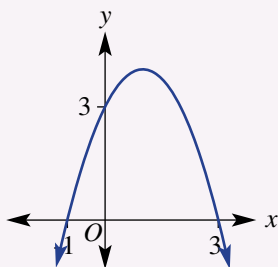
- a one x -intercept? b two x -intercepts? c no x -intercepts?

- 4 For what values of k does the graph of $y = 5x^2 + kx + 1$ have:

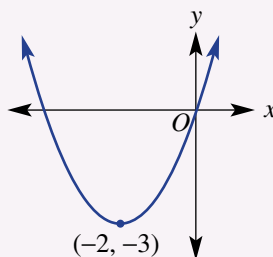
- a one x -intercept? b two x -intercepts? c no x -intercepts?

- 5 Find the rules for these parabolas.

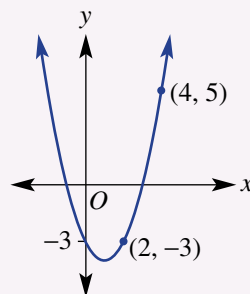
a



b



c



- 6 A graph of $y = ax^2 + bx + c$ passes through the points $A(0, -8)$, $B(-1, -3)$ and $C(1, -9)$. Use your knowledge of simultaneous equations to find the values of a , b and c and, hence, find the turning point for this parabola, stating the answer using fractions.
- 7 Determine the maximum vertical distance between these two parabolas at any given x -value between the points where they intersect:
 $y = x^2 + 3x - 2$ and
 $y = -x^2 - 5x + 10$
- 8 Two points, P and Q , are on the graph of $y = x^2 + x - 6$. The origin $(0, 0)$ is the midpoint of the line segment PQ . Determine the exact length of PQ .
- 9 A parabola, $y_1 = (x - 1)^2 + 2$, is reflected in the x -axis to become y_2 . Now y_1 and y_2 are each translated 2 units horizontally but in opposite directions, forming y_3 and y_4 . If $y_5 = y_3 + y_4$, sketch the graphs of the possible equations for y_5 .

Sketching $y = x^2 + bx + c$ e.g. $y = x^2 - 4x - 12$ y -intercept: $x = 0, y = -12$ x -intercepts: $y = 0, x^2 - 4x - 12 = 0$ Factorise $(x - 6)(x + 2) = 0$

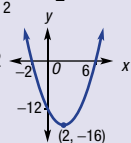
Null Factor Law

 $x - 6 = 0$ or $x + 2 = 0$ $x = 6$ or $x = -2$ Turning point: halfway between x -intercepts

$$x = \frac{6 + (-2)}{2} = 2$$

Substitute $x = 2$:

$$y = 2^2 - 4(2) - 12 = -16$$

 $(2, -16)$ minimum**Sketching $y = a(x - h)^2 + k$**

Any quadratic can be written in turning point form by completing the square.

e.g. sketch $y = (x - 1)^2 - 3$ Turning point $(1, -3)$ minimum y -intercept: $x = 0, y = 1 - 3$

$$= -2$$

 x -intercepts: $y = 0$

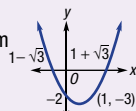
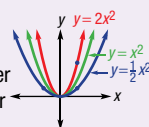
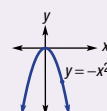
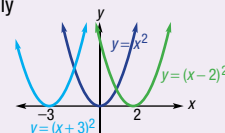
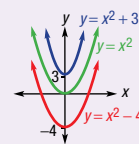
$$0 = (x - 1)^2 - 3$$

$$3 = (x - 1)^2$$

$$\pm\sqrt{3} = x - 1$$

$$\therefore x = 1 + \sqrt{3}, 1 - \sqrt{3}$$

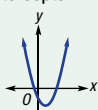
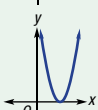
(alternatively use the Null Factor Law)

**Transformations and turning point form** $y = ax^2$ dilates graphcompared to $y = x^2$ $0 < a < 1$ graph is wider $a > 1$ graph is narrower $a < 0$ graph is inverted (reflected in x -axis) $y = (x - h)^2$ translates $y = x^2$ h units horizontally $h > 0$ move right $h < 0$ move left $y = x^2 + k$ translates $y = x^2$ k units vertically $k > 0$ move up $k < 0$ move down

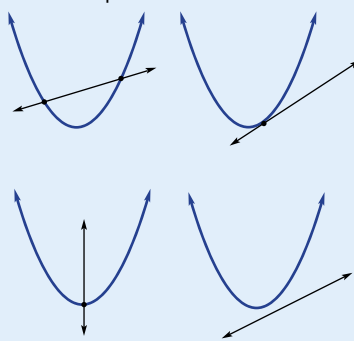
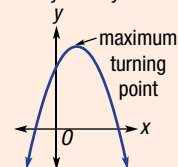
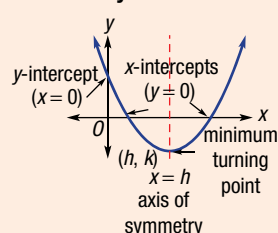
Turning point form

 $y = a(x - h)^2 + k$ has turning point at (h, k) **Quadratic formula**For $y = ax^2 + bx + c$, the quadratic formula can be used to find x -intercepts.i.e. if $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

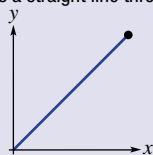
The discriminant $\Delta = b^2 - 4ac$ tells how many x -intercepts: $\Delta > 0$ two x -intercepts $\Delta = 0$ one x -intercept $\Delta < 0$ no x -intercepts**Parabolas, rates of change and variation****Lines and parabolas**

Using substitution we can solve simultaneously to find 0, 1 or 2 intersection points.

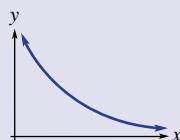
**Key features****Quadratic applications**Form a quadratic model based on the formulation in the question. Sketch the graph for suitable values of x , to locate max/min and intercepts.

Direct variation and inverse variationDirect variation: $y = kx$

- Both variables increase or decrease together.
- Graph is a straight line through $(0, 0)$.

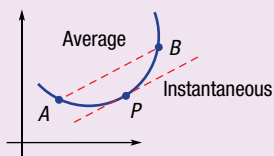
Inverse (indirect) variation: $y = \frac{k}{x}$ or $xy = k$

- If one variable increases, the other decreases.
- Graph is a hyperbola shape.

**Rates of change****Average and instantaneous rates of change**

The average rate of change between two points is equal to the gradient of the line joining the two points.

The instantaneous rate of change is equal to the gradient at a particular point and can be approximated by the average rate of change between two selected points.

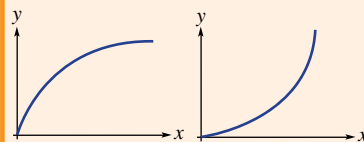
**Rates of change**

The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) vary.

Straight lines have a constant rate of change; curves have a varying rate of change.

Description of graphs:

- y is increasing or decreasing or constant.
- Rate of change is positive or negative or zero.
- Rate of change is increasing or decreasing or constant.



Positive decreasing gradient

Positive increasing gradient

Chapter checklist with success criteria

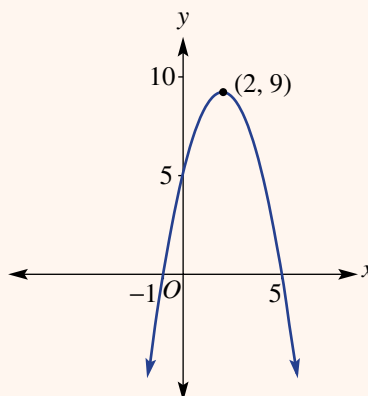
A printable version of this checklist is available in the Interactive Textbook



7A

1. I can identify the key features of a parabola.

e.g. For the graph shown determine the turning point (and whether it's a maximum or a minimum), the axis of symmetry, the x -intercepts and the y -intercept.

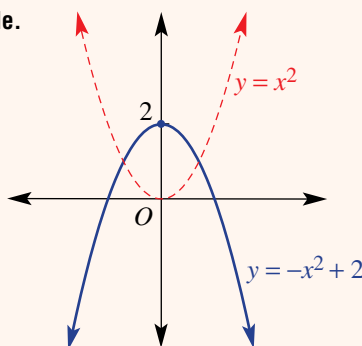


7A

2. I can find key features of transformed graphs from a rule.

e.g. Complete the table for the graph shown

Formula	Max or min	Turning point	y -value when $x = 1$
$y = -x^2 + 2$			



7B

3. I can sketch a quadratic relation involving a dilation or reflection.

e.g. Sketch the graph of $y = -4x^2$, labelling the turning point and one other point.



7B

4. I can sketch a quadratic relation involving translations.

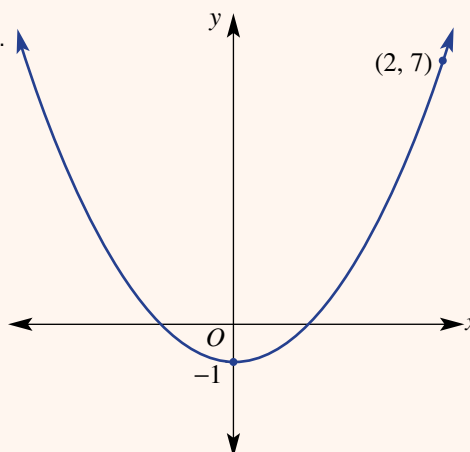
e.g. Sketch the graph of $y = (x - 1)^2 + 3$, labelling the turning point and y -intercept.



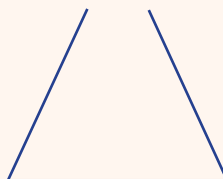
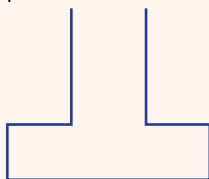
7B

5. I can find a rule in turning point form from a quadratic graph.

e.g. Find a rule for this parabola with turning point $(0, -1)$ and another point $(2, 7)$.



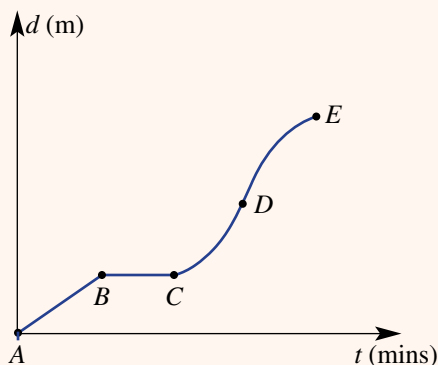
7C	6. I can use the x-intercepts to find the turning point and sketch a quadratic graph. e.g. Sketch the graph of $y = x^2 - 2x - 3$, labelling the intercepts and the turning point.	☐
7C	7. I can sketch a quadratic graph that is a perfect square. e.g. Sketch the graph of $y = x^2 - 4x + 4$, labelling the intercepts and turning point.	☐
7D	8. I can determine the key features of a graph in turning point form. e.g. For $y = 2(x - 1)^2 - 18$, determine the turning point, y -intercept and any x -intercepts.	☐
7D	9. I can sketch a quadratic graph by first completing the square. e.g. Sketch $y = x^2 + 8x + 20$ by first completing the square.	☐
7E	10. I can use the discriminant to find the number of x-intercepts of a quadratic graph. e.g. Determine the number of x -intercepts of the parabola given by $y = 2x^2 + 3x + 6$.	☐
7E	11. I can find the turning point of a quadratic using $x = -\frac{b}{2a}$. e.g. Determine the turning point coordinates of the parabola given by $y = 2x^2 + 8x - 5$.	☐
7E	12. I can sketch a quadratic graph using the quadratic formula. e.g. Sketch the graph of the quadratic $y = 3x^2 - 6x - 2$, labelling significant points. Round the x -intercepts to two decimal places.	☐
7F	13. I can apply quadratic models in word problems. e.g. A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let x cm be the breadth of the rectangle. i Use the perimeter to find an expression for the length of the rectangle in terms of x . ii Hence, find a rule for the area of the rectangle, $A \text{ cm}^2$, in terms of x and sketch its graph for suitable values of x . iii Use the graph to determine the maximum area that can be formed and the dimensions of the rectangle that give this area.	☐
7G	14. I can find the points of intersection of a line and a parabola. e.g. Find the points of intersection of $y = 2x^2 - 5$ and $y = x + 1$.	☐
7G	15. I can determine the number of points of intersection of a line and a parabola. e.g. Determine the number of solutions (points of intersection) of the equations $4x + y = -4$ and $y = x^2 + 2x + 5$.	☐
7H	16. I can graph height vs time to show the rate of change of height. e.g. Water is poured into these containers at a constant rate. Draw a graph for each showing the relationship between the height, h , of water in the container at time t . i ii	☐



7H

17. I can work with distance-time graphs.

e.g. The distance-time graph shows a journey by scooter over a number of minutes.



- Between what points does the graph show a speed which is constant?
- Between what points does the graph show a speed which is decreasing?
- Between point C and E, at which point is the scooter traveling the fastest?



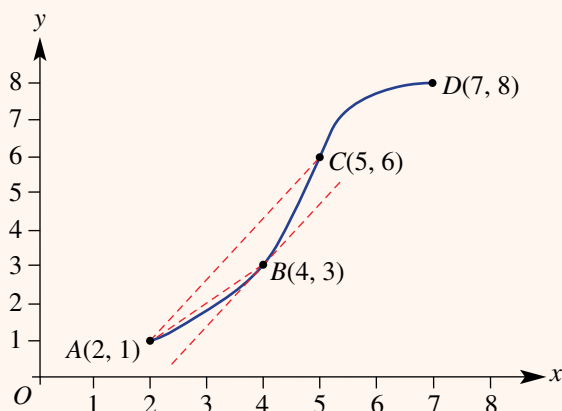
7I

18. I can find the average rate of change.e.g. Find the average rate of change of y in the rule $y = x^2 + 2$ as x changes from 1 to 3.

7I

19. I can approximate an instantaneous rate of change.

e.g. This graph shows part of a curve, with a tangent drawn at point B to illustrate the instantaneous rate of change at that point. Find the gradient of line segments AB and AC and state which one gives a better approximation of the instantaneous rate of change at point B.



7J

20. I can work with direct variation.e.g. If y is directly proportional to x and $y = 45$ when $x = 25$, determine the relationship between y and x and use this to find y when $x = 40$.

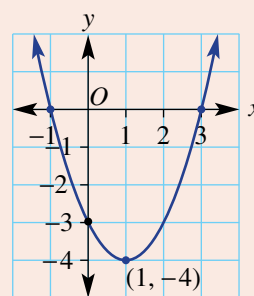
7J

21. I can work with inverse variation.e.g. If x and y are inversely proportional and $y = 5$ when $x = 4$, determine the constant of proportionality k and write the rule.

Short-answer questions

7A

- 1 State the following features of the quadratic graph shown.
- a turning point and whether it is a maximum or a minimum
 - b axis of symmetry
 - c coordinates of the x -intercepts
 - d coordinates of the y -intercept



7A/B

- 2 State whether the graphs of the following quadratics have a maximum or a minimum turning point and give its coordinates.

a $y = (x - 2)^2$

b $y = -x^2 + 5$

c $y = -(x + 1)^2 - 2$

d $y = 2(x - 3)^2 + 4$

7C

- 3 Sketch the quadratics below by first finding:

i the y -interceptii the x -intercepts, using factorisation

iii the turning point

a $y = x^2 - 4$

b $y = x^2 + 8x + 16$

c $y = x^2 - 2x - 8$

7D

- 4 Complete the following for each quadratic below.

i State the coordinates of the turning point and whether it is a maximum or a minimum.

ii Find the coordinates of the y -intercept.iii Find the coordinates of the x -intercepts (if any).

iv Sketch the graph, labelling the features above.

a $y = -(x - 1)^2 - 3$

b $y = 2(x + 3)^2 - 8$

7D

- 5 Sketch the following quadratics by completing the square. Label all key features with exact coordinates.

a $y = x^2 - 4x + 1$

b $y = x^2 + 2x + 6$

c $y = x^2 + 3x - 2$

7D/E

- 6 State the number of x -intercepts of the following quadratics either by using the discriminant or by inspection where applicable.

a $y = (x + 4)^2$

b $y = (x - 2)^2 + 5$

c $y = x^2 - 2x - 5$

d $y = 2x^2 + 3x + 4$

7E

- 7 For the following quadratics:

i find the coordinates of the y -intercept.ii use $x = -\frac{b}{2a}$ to find the coordinates of the turning point.iii use the quadratic formula to find the x coordinates of the x -intercepts, rounding to one decimal place.

iv sketch the graph.

a $y = 2x^2 - 8x + 5$

b $y = -x^2 + 3x + 4$



7G

8 Solve these equations simultaneously.

$$\begin{aligned} \text{a } y &= x^2 + 4x - 2 \\ y &= 10 \end{aligned}$$

$$\begin{aligned} \text{b } y &= 2x^2 + 5x + 9 \\ y &= -x + 4 \end{aligned}$$

$$\begin{aligned} \text{c } y &= x^2 + 1 \\ 2x + 3y &= 4 \end{aligned}$$

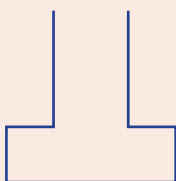
7G

9 Use the discriminant to show that the line $y = x + 4$ intersects the parabola $y = x^2 - x + 5$ in just one place.

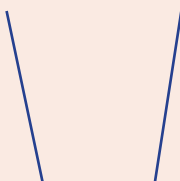
7H

10 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h , of water in the container at time t .

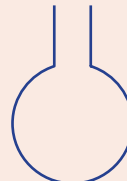
a



b

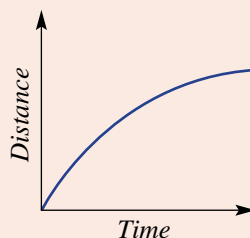


c



7H

11 For the distance–time graph showing a journey from home, describe the journey in relation to distance from home, gradient of graph and speed.



7I

12 For the relations with rules below, find the average rate of change of y as x changes from

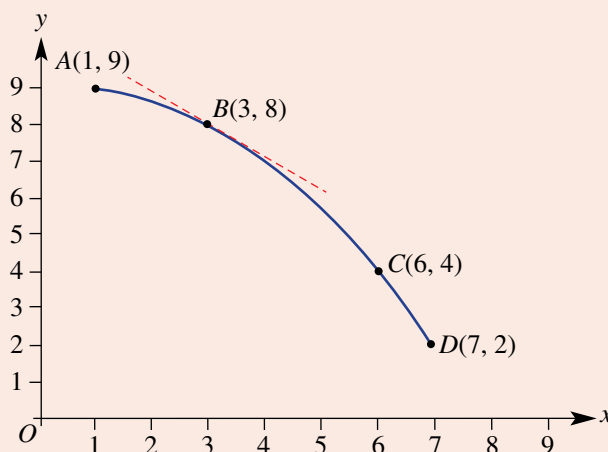
i 0 to 2

ii -1 to 4

$$\text{a } y = x^2 + 2x$$

$$\text{b } y = -x^2 - 2x + 3$$

7I

13 This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at the point. Use the points A to D to find the gradient of the line segment that gives the best approximation of the instantaneous rate of change at point B .

7J

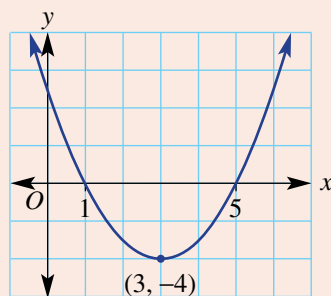
- 14 a If y varies directly with x and $y = 10$ when $x = 2$, find the rule linking y with x .
 b If $y = \frac{k}{x}$ and $x = 6$ when $y = 12$, determine:
 i the value of k ii y when $x = 4$ iii x when $y = 0.7$

Multiple-choice questions

7A

- 1 The equation of the axis of symmetry of the graph shown is:

A $y = -4$
 B $x = 3$
 C $x = -4$
 D $y = 3$
 E $y = 3x$



7B

- 2 Compared to the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is:

A translated 3 units down B translated 3 units left
 C dilated by a factor of 3 D translated 3 units right
 E translated 3 units up

7B

- 3 The coordinates and type of turning point of $y = -(x + 2)^2 + 1$ is:

A a minimum at $(-2, -1)$ B a maximum at $(2, 1)$
 C a minimum at $(2, 1)$ D a maximum at $(2, -1)$
 E a maximum at $(-2, 1)$

7B

- 4 The y -intercept of $y = 3(x - 1)^2 + 4$ has coordinates:

A $(0, 1)$ B $(0, 4)$ C $(0, \frac{1}{3})$ D $(0, 7)$ E $(0, 3)$

7C

- 5 The x -intercept(s) of the graph of $y = x^2 + 3x - 10$ have x -coordinates at x equals:

A 2, -5 B -10 C 5, -2
 D -5, -2 E 5, 2

7C

- 6 A quadratic graph has x -intercepts with x -coordinates at $x = -7$ and $x = 2$. The x -coordinate of the turning point is:

A $x = -\frac{5}{2}$ B $x = -\frac{7}{2}$ C $x = \frac{9}{2}$
 D $x = \frac{5}{2}$ E $x = -\frac{9}{2}$

7D

- 7 The quadratic rule $y = x^2 - 4x - 3$, when written in turning point form, is:

A $y = (x - 2)^2 - 3$ B $y = (x - 4)^2 + 1$ C $y = (x - 2)^2 - 7$
 D $y = (x + 4)^2 - 19$ E $y = (x + 2)^2 - 1$

7E

- 8 A quadratic graph $y = ax^2 + bx + c$ has two x -intercepts. This tells us that:

A The graph has a maximum turning point.
 B $-\frac{b}{2a} < 0$
 C There is no y -intercept.
 D $b^2 - 4ac > 0$
 E $b^2 - 4ac = 0$

7F

- 9 A toy rocket follows the path given by $h = -t^2 + 4t + 6$, where h is the height above ground, in metres, t seconds after launch. The maximum height reached by the rocket is:

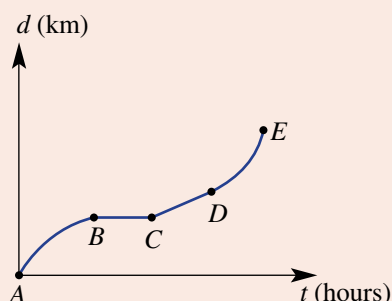
A 10 metres B 2 metres C 6 metres
D 8 metres E 9 metres

7H

- 10 The distance–time graph shows a journey by bicycle over a number of hours.

Between what points does the graph show a speed which is increasing?

A A to B
B C to D
C D to E
D C to D and D to E
E C to D and A to B



7I

- 11 The average rate of change of y as x changes from 1 to a ($a > 1$) in the rule $y = x^2 + 2$ is 4. The value of a is:

A 2 B 3 C 4 D 5 E 6

7J

- 12 If y is inversely proportional to x , the equation is of the form:

A $y = kx$ B $y = kx + c$ C $y = \frac{x}{k}$
D $y = kx^2$ E $y = \frac{k}{x}$

Extended-response questions

- The cable for a suspension bridge is modelled by the equation $h = \frac{1}{800}(x - 200)^2 + 30$, where h metres is the distance above the base of the bridge and x metres is the distance from the left side of the bridge.
 - Determine the turning point of the graph of the equation.
 - If the bridge is symmetrical, determine the suitable values of x .
 - Determine the range of values of h .
 - Sketch a graph of the equation for the suitable values of x .
 - What horizontal distance does the cable span?
 - What is the closest distance of the cable from the base of the bridge?
 - What is the greatest distance of the cable from the base of the bridge?
- 200 metres of fencing is to be used to form a rectangular paddock. Let x metres be the breadth of the paddock.
 - Write an expression for the length of the paddock in terms of x .
 - Write an equation for the area of the paddock ($A \text{ m}^2$) in terms of x .
 - Decide on the suitable values of x .
 - Sketch the graph of A versus x for suitable values of x .
 - Use the graph to determine the maximum paddock area that can be formed.
 - What will be the dimensions of the paddock to achieve its maximum area?